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**The Tensor Structure of Constitutive Equations  
for Linear Atmospheric Heat and Momentum Exchange  
with Axisymmetric Coefficients**

by  
Fritz Herbert  
(with 3 figures and 3 tables in the text)

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Anschrift des Verfassers:

Dr. Fritz Herbert, Institut für Meteorologie, Johannes Gutenberg-Universität, 6500 Mainz

## Abstract

Small-scale heat and momentum exchange processes were linearly parameterized using the assumption that the coefficients which couple fluxes and driving forces in the linear regression laws are symmetrical with respect to an invariable space direction.

To carry out this analysis the atmosphere (ocean) was considered as a one-component viscous fluid not subjected to external forces. As fundamental physical concepts we applied, in addition to the conservation laws of energy and momentum the principle of conservation of angular momentum. This concept plays an important role whenever momentum and energy transport involves not only degrees of freedom of translatory but also of rotational motions so that an antisymmetric component of the stress tensor must be considered. A systematic parameterization shows that transport processes due to the antisymmetric stress part, become relevant which were so far ignored in meteorological investigations.

With the assumption of axisymmetric exchange coefficients, the regression laws simplify considerably since the number of independent coefficients is greatly reduced. Thus, in particular, some cross-processes cancel out. From the extended regression laws derived in this analysis follow additionally, as special cases, the well-known linear laws of isotropic systems (Curie-theorem) as well as the flux-gradient relationships usually hypothesized to planetary boundary layer exchange.

## Zusammenfassung

Anhand des Wärme- und Impulstransports behandelten wir die lineare Parametrisation von kleinräumigen Austauschprozessen, wobei die Koeffizienten, welche Flüsse und Antriebe in den linearen Regressionsgesetzen verknüpfen, symmetrisch in bezug auf eine invariante Raumrichtung angenommen wurden.

Dazu betrachten wir die Atmosphäre (Ozean) als eine ein-komponentige viskose Flüssigkeit ohne den Einfluß externer Kräfte. Als grundlegende Erhaltungssätze wurden außer dem Energie- und Impulssatz auch der Drehimpulssatz verwendet. Er spielt dann eine wesentliche Rolle, wenn der Drucktensor auch einen antisymmetrischen Teil enthält, der dadurch bedingt ist, daß im Impuls- und Energietransport nicht nur die Freiheitsgrade der translatorischen, sondern auch der rotatorischen Bewegung zu berücksichtigen sind. Eine systematische Parametrisation zeigt dann, daß durch den antisymmetrischen Anteil des Drucktensors Transportprozesse relevant werden, die in meteorologischen Betrachtungen bisher unbeachtet blieben.

Mit der Annahme, daß die Tensoren der Austauschkoefizienten achsensymmetrisch sind, vereinfachen sich die Regressionsgesetze erheblich, wobei die Zahl der unabhängigen Koeffizienten stark reduziert wird; insbesondere werden eine Reihe von Oberlagerungsprozessen eliminiert. Aus den erweiterten Regressionsgesetzen dieser Analyse ergeben sich außerdem als Spezialfälle die linearen Gesetze isotroper Systeme (Curie-Theorem) sowie die Fluß-Gradient-Beziehungen wie sie gewöhnlich für den Austausch in der planetarischen Grenzschicht unterstellt werden.

## 1 Introduction

### a General Problem

In the present article emphasis will be placed on the development of the basic concepts and equations that are essential for special studies of atmospheric (and oceanic) small subgrid-scale transport processes. One of the physical problems, associated with the description of atmospheric states and developments by grid representation, is the treatment of those processes in the scale smaller than grid size. Despite the smallness of this scale the exchange processes perform work and transport considerable quantities of water vapor, momentum and heat; thus, the energy content coupled to these small-scale processes is quite significant.

To gain an insight in depths of the various participating tensor type exchange processes, we shall first briefly reexamine the theory of linear parameterization following the classical Onsager-Casimir approach which is based on the second law of thermodynamics. Van Mieghem (1949) was the first to direct attention to the particular value of this theory for meteorological application. He applied the theory to discuss irreversible subgrid-scale physics in an attempt to predict atmospheric large scale states. More recently, the same approach was used by Yositate (1956), Suzuki (1957) as well as by Hinkelmann (1973) who also emphasized that theory and practical simulation of long range weather forecasting and general circulation could not be effectively promoted without sufficient understanding of the irreversible phenomena.

The above mentioned method of parameterization implied by the Onsager-Casimir theory shows clearly (e.g. de Groot and Mazur, 1969; Gyarmati, 1970) that the irreversible processes are associated to the conservation laws of mass, energy, linear and angular momentum. It should be emphasized that the conservation law of angular momentum so far has hardly ever been considered in studying small scale atmospheric

(and oceanic) transport processes. The only consequence derived from this conservation law was at most the symmetry properties of the stress tensor which follow from the assumption that the external mechanical angular momentum  $\vec{r} \times \rho \vec{u}$  ( $\vec{r}$  = radius vector,  $\rho$  = density,  $\vec{u}$  = velocity) was accepted as conservative quantity. Moreover we are familiar with internal angular momenta which stem from rotational motions of molecules and especially from turbulent vortices which, on the average, cause an intrinsic macroscopic angular momentum. For this reason, in the treatment of the stress tensor and the velocity field, it is mandatory to involve the exchange of external and intrinsic angular momentum.

### b Specific Treatment

We shall first concentrate on processes which are related to the stress-tensor and the velocity field. In this discussion we shall include rotational phenomena which come from the exchange of mechanically external and intrinsic angular momentum. This leads to a discussion of the antisymmetric part of the stress tensor as well as the balances of angular-momentum and intrinsic rotational energy (according to Grad, 1952 or Gyarmati, 1970) which have not yet been considered with sufficient attention in studies of atmospheric and oceanic exchange. Surely, the conservation law of angular momentum plays an important part in atmospheric and oceanic investigations when irreversible transfer processes are examined in the presence of rotational motions of the fluid. For the description of such processes we rely on the work of Grad and Gyarmati as well as on Meixner's (1961) treatise which allows to insert the exchange of intrinsic and external angular momentum into the thermodynamics of irreversible processes.

When the theory of irreversible processes for continuous systems is extended to the turbulent transfer of energy, momentum, angular-momentum and mass it must be regarded as a characteris-

tic property that the exchange coefficients are not isotropic. The non-isotropic behaviour is obvious, for example, since one considers in almost all cases of thermal stratification the fact that the principal axes of the stress tensor of turbulent friction (represented by the Reynolds-stress formulation) and the corresponding strain tensor are not aligned (see e. g. Monin and Yaglom, 1971, and Fiedler, 1975). As a realistic symmetry condition for the tensorial structure of the coefficients the condition of local axisymmetry should be introduced, where, in a first order approach the vertical direction may be regarded as the axis of symmetry. This corresponds to observations and experiments in wind channels which show that small-scale turbulent transports have considerably different length scales along vertical and horizontal directions.

The main object of the present article is to derive the axisymmetric expressions of all tensor types of exchange coefficients which may be of interest for the description of small-scale irreversible processes. Above all, axisymmetric coefficients should be taken into account together with linear constitutive equations to treat eddy-fluxes most adequately in large-scale atmospheric and oceanic motions.

It is not our intent in this article to examine special effects described by these equations or to determine the constitutive coefficients in dependence on the static stability of stratification, on the baroclinic stability, or on other characteristic properties of the fluid and motion. A complete discussion of these topics is not yet possible.

A suitable method to develop axisymmetric coefficient tensors including all natural symmetries of the constitutive equations is obtainable by means of the relevant theorems of the theory of invariants. We shall establish the axisymmetric equations similar to the procedure of Robertson (1940), Batchelor (1946) and Chandrasekhar (1950) who make use of the invariance theory in the treatment of the so-called correlation method of isotropic and axisymmetric turbulence (see also Panchev, 1971). But it should be clarified that in contrast to the correlation-method, where the conditions of axisymmetry is applied to the irreversible fluxes, in the present discussion this condition is applied to the coefficients of the constitutive equations given in section 3. From this follows that turbulence characteristics such as the irreversible fluxes, should in no way be considered as axial symmetrical.

## 2 The Basic Laws of Momentum Angular Momentum and Energy

As far as the tensor structure of conductivity or exchange coefficients is concerned it is sufficient to develop only one coefficient of each tensor class since coefficients of the same tensor character possess the same structure independent of their physical meaning (e.g. de Groot and Mazur, 1969). According to the aim of this article we may thus confine ourselves to considering as atmospheric (or oceanic) system a simple one-component fluid which we assume to be not subjected to any external forces, and for which we must only consider the irreversible transport processes due to friction and heat exchange phenomena. In order to discuss this problem as complete as possible we must go back to the conservation laws of energy and momentum of the fluid. In the absence of external bulk-forces the balances of these laws can be formulated, with respect to unit volume, as follows

$$\rho \frac{d\epsilon}{dt} + \nabla \cdot \vec{I}_\epsilon = 0 \quad (2-1)$$

$$\rho \frac{d\vec{u}}{dt} + \nabla \cdot \mathbf{P} = 0 \quad (2-2)$$

where the vector  $\vec{I}_\epsilon$  characterizes, at first quite formally, the non-convective flow per unit mass of the total energy and where the dyad  $\mathbf{P}$  represents the total stress tensor per unit mass due to the isotropic pressure force as well as the friction forces.<sup>†</sup>) In contrast to what is usually done we will not assume that  $\mathbf{P}$  is symmetric, but leave principally open the possibility that  $\mathbf{P}$  contains an antisymmetric part. Then, in order to obtain a complete balance of momentum, we need to consider the law of conservation of total angular momentum  $\vec{Q}$ . According to Grad (1952)  $\vec{Q}$  can be considered as the sum of two

contributions

$$\left. \begin{aligned} \vec{Q} &= \vec{R} + \vec{S} \quad , \quad \text{with} \\ \vec{R} &= \vec{r} \times \vec{u} \quad , \quad \text{and} \\ \vec{S} &= \Theta \vec{\omega} \end{aligned} \right\} \quad (2-3)$$

Therein, the quantities  $\vec{Q}$ ,  $\vec{R}$  and  $\vec{S}$  are axial vectors which, also, could alternatively be written as second order antisymmetric tensors; physically,  $\vec{R}$  denotes the external angular momentum per unit mass given by the vector-product of the mean velocity  $\vec{u}$  and the position vector  $\vec{r}$  both related to an arbitrary system of coordinates, while  $\vec{S}$  denotes the internal angular momentum given as the product of the inertia moment per unit mass  $\Theta$  and the angular velocity  $\vec{\omega}$  which both arise as the consequence of a possible internal rotation of the constituent particles of the system. As a result of these properties the balance equations for the angular momenta can be written (see Appendix A) after assuming the intrinsic flux-density to be negligible, in the form:

$$\begin{aligned} \rho \frac{d\vec{Q}}{dt} + \nabla \cdot (\vec{r} \times \mathbf{P}) &= 0 \\ \rho \frac{d\vec{R}}{dt} + \nabla \cdot (\vec{r} \times \mathbf{P}) &= 2\vec{P}^a \quad (2-4) \\ \rho \frac{d\vec{S}}{dt} &= -2\vec{P}^a \end{aligned}$$

which has been quoted firstly by Grad (1952) and later for example by de Groot and Mazur (1969) as well as Gyarmati (1970). The quantity  $\vec{P}^a$  denotes the axial stress vector which was introduced instead of the antisymmetric part of the stress tensor  $\mathbf{P}^{as} = \frac{1}{2}(\mathbf{P} - \tilde{\mathbf{P}})$ ; the components of  $\vec{P}^a$  and  $\mathbf{P}^{as}$  are related to one another according to the formula

$$\begin{aligned} P_k^a &= \frac{1}{2} (P_{ij} - P_{ji}) = -\frac{1}{2} (P_{ji} - P_{ij}) \\ i, j, k &= 1, 2, 3 \quad (\text{cycl.}) \end{aligned} \quad (2-5)$$

It is clear from Eq. (2-4) that, if the fluid

<sup>†</sup>) For identifying the symbols used in the present text the reader is referred to the list of symbols at the end of this article.

possesses no intrinsic internal motion, i. e.  $\vec{\omega} = 0$  and  $\vec{S} = 0$ , then  $P$  is symmetric:

$$\vec{P}^a = 0, \text{ i.e. } P = \tilde{P} \quad (2-6)$$

and  $\vec{R}$ , which is conserved, equals the total angular momentum. Hence,

$$\rho \frac{d\vec{R}}{dt} + \nabla \cdot (\vec{r} \times P) = 0 \text{ with } P = \tilde{P} \quad (2-7)$$

On the other hand we can write, if the stress tensor is symmetric,

$$\frac{d\vec{S}}{dt} = 0 \text{ and } \rho \frac{d\vec{R}}{dt} + \nabla \cdot (\vec{r} \times P) = 0 \quad (2-8)$$

which means that the external and internal angular momentum are separately conserved.

In atmospheric investigations this condition applies to eddy exchange by small-scale turbulence where it is assumed that the friction processes can be expressed in terms of the symmetric stress-tensor  $\rho \vec{u}'' \vec{u}''$ . This tensor is given by the statistical correlation between the velocity oscillations  $\vec{u}''$  and the momentum oscillations  $\rho \vec{u}''$  from which the eddy-kinetic energy per unit volume  $E = \frac{1}{2} \rho \vec{u}''^2$  can be formed. In so doing, the total stress of the turbulent motion may be expressed by

$$P_t = \overline{\rho \vec{u}'' \vec{u}''} - \frac{2}{3} E \delta + p \delta, \quad (2-9)$$

i.e.  $\vec{P}_t^a = 0$

Equation (2-9) requires that, in accordance with Eq. (2-8), both  $\vec{R}_t$  and  $\vec{S}_t$  are conserved. Thus, it is obvious from Eqs. (2-1,2) and Eq. (2-4) that with use of the stress-tensor  $\rho \vec{u}'' \vec{u}''$  intrinsic rotations of the elements of turbulent motion as well as the external angular momentum  $\vec{r} \times \vec{u}$  remain completely disregarded. In order to prevent those strong restrictions we would provide instead of Eq. (2-9), as a more realistic formulation of the eddy stress tensor,

$$P_t = p \delta + (P_t^s - p \delta) + P_t^{as} \quad (2-10)$$

which is relevant if rotational phenomena are to be taken into consideration. Note then that

the expression  $\overline{\rho \vec{u}'' \vec{u}''} - \frac{2}{3} E \delta$  (Eq. 2-8) following from the Reynolds-method of taking the mean of the momentum balance equation is no suitable approach to the symmetric component  $P_t^s - p \delta$  if the antisymmetric stress component  $P_t^{as}$  plays a role in the total friction stress.

For the energy budget we also must consider the angular momentum  $\Theta \vec{\omega}$  (see Eq. 2-2) from which we may form the intrinsic rotational energy

$$E_R = \frac{1}{2} \Theta \omega^2 \quad (2-11)$$

Also, it is clear that, due to our basic assumptions, potential energies and energies due to diffusion processes are excluded. Thus, let us decompose the total energy per unit mass into the three parts:

$$E = E_T + E_R + E_I \quad (2-12)$$

where  $E_T = \frac{1}{2} \vec{u}^2$  is the kinetic energy and  $E_I$  the internal energy, both per unit mass. We can find the balance equation for  $E_R$  from Eq. (2-4) by scalar multiplication of the  $\vec{S}$ -balance equation with  $\vec{\omega}$  as:

$$\rho \frac{dE_R}{dt} = -2 \vec{\omega} \cdot P^a = -2 \omega : P^{as} \quad (2-13)$$

Therein,  $\omega$  is the antisymmetric tensor conjugated to  $\vec{\omega}$ , the components of which are given by  $\omega_{ij} = -\omega_{ji} = \omega_k$  where  $i, j, k = 1, 2, 3$ , cycl. The balance equation for the kinetic energy (see Appendix A) follows by means of a scalar multiplication of the equation of motion (2-2) with  $\vec{u}$  as

$$\rho \frac{dE_T}{dt} = -\vec{u} \cdot \nabla \cdot P = -\nabla \cdot (P \cdot \vec{u}) + \tilde{P} : \nabla \vec{u} \quad (2-14)$$

Subtracting Eq. (2-13) and Eq. (2-14) from the energy conservation law Eq. (2-1) we obtain the balance equation of internal energy

$$\rho \frac{dE_I}{dt} = -\nabla \cdot (\vec{I}_E - P \cdot \vec{u}) - \tilde{P} : \nabla \vec{u} + 2 \vec{P}^a \cdot \vec{\omega} \quad (2-15)$$

which is identical with the first law of thermodynamics. Thus, the corresponding heat flux density can be expressed by:

$$\vec{I}_h = \vec{I}_\epsilon - P \cdot \vec{u} \quad (2-16)$$

including the total external heat transport supplied by reversible and irreversible exchange. With Eq. (2-16) the internal energy balance Eq. (2-15) assumes the form

$$\begin{aligned} \rho \frac{d\epsilon_I}{dt} &= -\nabla \cdot \vec{I}_h - \tilde{P} : \nabla \vec{u} + 2\tilde{P}^a \cdot \vec{\omega} \\ &= -\nabla \cdot \vec{I}_h - \tilde{P} : \nabla \vec{u} + 2P^{as} : \omega \end{aligned} \quad (2-17)$$

where the expressions  $-\tilde{P} : \nabla \vec{u}$  and  $2P^{as} : \omega$  (or alternatively,  $2\tilde{P}^a \cdot \vec{\omega}$ ) indicate the total reversible and irreversible working capacity due to the stress-forces. Also, we must consider that additional terms representing reversible as well as irreversible work, the latter being especially caused by diffusive effects, do not occur because of the simplifications made at the beginning of this section.

For further treatment it turns out to be useful to split up these "work-terms" into their reversible and irreversible parts. Let us split therefore the total stress tensor into the isotropic equilibrium pressure  $p$ , multiplied by the unit tensor  $\delta$ , and the friction stress  $I$ , i.e.

$$P = p\delta + I \quad (2-18)$$

Furthermore it is useful to separate  $I$  additively into such terms which differ through their invariance properties. Thus we proceed to

$$I = f\delta + I^{0,s} + I^{as} \quad (2-19)$$

where  $f$  is a third of the trace,  $I^{0,s}$  is the symmetrical part with zero trace and  $I^{as}$  is the antisymmetric part of  $I$ . Using the previous conditions it follows consequently that  $I^{as} = P^{as}$ , or in the equivalent vector formulation  $\vec{I}^a = \vec{P}^a$ , as well. In analogy to Eq. (2-19) let us also split up the velocity gradient in

terms of the components

$$\left. \begin{aligned} \nabla \vec{u} &= \frac{1}{3} \text{Tr}(\nabla \vec{u})\delta + (\nabla \vec{u})^{0,s} + \\ &\quad + (\nabla \vec{u})^{as}, \\ \text{where } \text{Tr}(\nabla \vec{u}) &= \nabla \cdot \vec{u}, \\ (\nabla \vec{u})^{0,s} &= \frac{1}{2}(\nabla \vec{u} + \vec{u} \nabla) \\ &\quad - \frac{1}{3} \nabla \cdot \vec{u} \delta, \\ (\nabla \vec{u})^{as} &= \frac{1}{2}(\nabla \vec{u} - \vec{u} \nabla) \end{aligned} \right\} \quad (2-20)$$

which imply the vectorial invariants

$$(\nabla \vec{u})_X^{0,s} = 0, \quad (\nabla \vec{u})_X^{as} = \nabla \times \vec{u} \quad (2-21)$$

Inserting Eqs. (2-18) to (2-21) into Eq. (2-17) the balance equation of the internal energy then may be expressed through

$$\begin{aligned} \rho \frac{d\epsilon_I}{dt} &= -(p+f)\nabla \cdot \vec{u} - I^{0,s} : (\nabla \vec{u})^{0,s} - \\ &\quad - \vec{I}^a \cdot (\nabla \times \vec{u} - 2\vec{\omega}) - \nabla \cdot \vec{I}_h \end{aligned} \quad (2-22)$$

or, on account of the law of mass conservation, i.e.  $\nabla \cdot \vec{u} = \rho^{-1} d\rho/dt$  in the equivalent form

$$\begin{aligned} \rho \left( \frac{d\epsilon_I}{dt} + p \frac{d(1/\rho)}{dt} \right) + f \nabla \cdot \vec{u} + I^{0,s} : (\nabla \vec{u})^{0,s} + \\ + \vec{I}^a \cdot (\nabla \times \vec{u} - 2\vec{\omega}) + \nabla \cdot \vec{I}_h = 0 \end{aligned} \quad (2-23)$$

In Eq. (2-23) it becomes evident that the anti-symmetric part of the stress tensor which, according to Eq. (2-4), affects the transformation of internal and external angular momentum, contributes to energy dissipation<sup>†</sup>

<sup>†</sup> Grad (1952), Meixner (1961), and particularly Baranowski and Romotowski (1964) elaborated on the dissipating character of  $\vec{I}^a$ .

### 3 The Governing Constitutive Laws for the Transport Processes

#### a The Balance Of Entropy And The Linear Regression Relations

To develop the balance of entropy we apply the second law of thermodynamics in the form of the Gibbs basic equation (Meixner and Reik, 1959; Hasse, 1963) which reads, when mass changes are not taken into consideration,

$$T \frac{ds}{dt} = \frac{d\epsilon_I}{dt} + p \frac{d^1s}{dt} \quad (3-1)$$

According to the classical theory of irreversible processes of weak non-equilibrium oscillations (elaborated on by de Groot and Mazur, 1969; Meixner, 1961, 1968; Gyarmati, 1970 where discussions of angular momentum effects are included) we can insert Eq. (2-23) into Eq. (3-1). In carrying out the procedure of balancing we thus proceed to the entropy equation

$$\begin{aligned} \rho \frac{ds}{dt} + \nabla \cdot \left( \frac{\vec{I}_h}{T} \right) &= - \vec{I}_h \cdot \frac{\nabla T}{T^2} - \\ &- \int \frac{\nabla \cdot \vec{u}}{T} - I^{0,s} : \frac{(\nabla \vec{u})^{0,s}}{T} - \\ &- \vec{I}^a \cdot (\nabla \times \vec{u} - 2\vec{\omega}) \geq 0 \end{aligned} \quad (3-2)$$

according to which the dissipation function due to the total entropy production may be formulated as:

$$\begin{aligned} T\sigma &= - \vec{I}_h \cdot \frac{\nabla T}{T} - \\ &- \int \nabla \cdot \vec{u} - I^{0,s} : (\nabla \vec{u})^{0,s} - \\ &- \vec{I}^a \cdot (\nabla \times \vec{u} - 2\vec{\omega}) \geq 0 \end{aligned} \quad (3-3)$$

On the right side of Eq. (3-3) we recognize a sum of products consisting of scalars, polar and axial vectors and of symmetric tensors of 2nd order without trace. Note in Eq. (3-3) that the thermodynamic driving forces of various tensor character being the factors of the irreversible fluxes discussed in the previous section, are to be considered as available quantities.

According to the linear Onsager-theory, which implies weak non-equilibrium, the fluxes are proportional to these forces. However in formulating this proportionality we must consider that, in accordance with the condition  $\sigma = 0$  for thermodynamic equilibrium, all fluxes and forces disappear simultaneously, i.e.

$$\begin{aligned} \vec{I}_h &= 0, \quad \int = 0, \quad I^{0,s} = 0, \\ \vec{I}^a &= 0, \quad \text{if} \\ \nabla T &= 0, \quad \nabla \cdot \vec{u} = 0, \quad (\nabla \vec{u})^{0,s} = 0, \\ \nabla \times \vec{u} &= 0, \quad \vec{\omega} = 0 \end{aligned} \quad (3-4)$$

Hence, each cartesian flux-component depends, in the absence of any natural conditions of symmetry, on all cartesian force components in terms of linear and homogeneous regression laws. Following the general treatment of Gyarmati (1970) the most general dependence between the flux and force tensors of Eq. (3-3) can be expressed in the form

$$\begin{aligned} \int &= -K^{ss} \nabla \cdot \vec{u} - \vec{K}^{sv} \cdot \nabla T - \\ &- \vec{K}^{sa} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) - K^{st} : (\nabla \vec{u})^{0,s} \end{aligned} \quad (3-5)$$

$$\begin{aligned} \vec{I}_h &= -\vec{K}^{vs} \nabla \cdot \vec{u} - K^{vv} \cdot \nabla T - \\ &- K^{va} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) - K^{vt} : (\nabla \vec{u})^{0,s} \end{aligned} \quad (3-6)$$

$$\begin{aligned} \vec{I}^a &= -\vec{K}^{as} \nabla \cdot \vec{u} - K^{av} \cdot \nabla T - \\ &- K^{aa} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) - K^{at} : (\nabla \vec{u})^{0,s} \end{aligned} \quad (3-7)$$

$$\begin{aligned} I^{0,s} &= -K^{ts} \nabla \cdot \vec{u} - K^{tv} \cdot \nabla T - \\ &- K^{ta} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) - K^{tt} : (\nabla \vec{u})^{0,s} \end{aligned} \quad (3-8)$$

Therein the  $K$  - coefficients correspond to the conductivity or exchange coefficients of heat and friction and are marked by upper indices indicating the tensorial character of their connected fluxes and forces. Hence, in general we have to distinguish linear combinations of the following tensor types:

$S$  = scalar,  $V$  = polar vector,  $a$  = axial

vector,  $t$  = symmetric tensor without trace, so that the coefficients occurring in the linear constitutive equations for heat and friction transports must apparently be tensors of the order 0, 1, 2, 3 and 4 and of polar or axial character indicated in the following table

| tensor order | 0        | 1                            | 2                                | 3                | 4        |
|--------------|----------|------------------------------|----------------------------------|------------------|----------|
| polar tensor | $K^{SS}$ | $\vec{K}^{SV}, \vec{K}^{VS}$ | $K^{VV}, K^{aa}, K^{St}, K^{Ts}$ | $K^{Vt}, K^{tV}$ | $K^{tt}$ |
| axial tensor | -        | $\vec{K}^{Sa}, \vec{K}^{As}$ | $K^{Va}, K^{aV}$                 | $K^{at}, K^{ta}$ | -        |

Table 1 : Tensor order and character of the exchange coefficients

Note that the coefficient tensors  $K^{St}, K^{Vt}, K^{at}$  and  $K^{tt}$  must be symmetric and have a zero trace in their last pair of cartesian indices since  $(\text{Grad } \vec{u})^{0,S}$  is a symmetric tensor with zero trace. Subsequently, the tensors  $K^{ts}, K^{tV}, K^{ta}$  and  $K^{tt}$  must be symmetric with zero trace with respect to their first pair of indices since the friction flux  $\vec{I}^{0,S}$  is a symmetric tensor without trace. Hence, there are the fundamental symmetry conditions

$$K_{ij}^{St} = K_{ji}^{St}, \quad K_{ij}^{ts} = K_{ji}^{ts} \quad (3-9)$$

$$\sum_i K_{ii}^{St} = 0, \quad \sum_i K_{ii}^{ts} = 0$$

$$K_{ijk}^{Vt} = K_{ikj}^{Vt}, \quad K_{ijk}^{tV} = K_{jik}^{tV} \quad (3-10)$$

$$\sum_j K_{ijj}^{Vt} = 0, \quad \sum_i K_{iik}^{tV} = 0$$

$$K_{ijk}^{at} = K_{ikj}^{at}, \quad K_{ijk}^{ta} = K_{jik}^{ta} \quad (3-11)$$

$$\sum_j K_{ijj}^{at} = 0, \quad \sum_i K_{iik}^{ta} = 0$$

$$K_{ijkl}^{tt} = K_{jikl}^{tt} = K_{ijlk}^{tt}$$

$$\sum_i K_{iike}^{tt} = 0 \quad (3-12)$$

$$\sum_k K_{ijkk}^{tt} = 0$$

Note that the polar tensors  $K^{aa}$  and  $K^{VV}$  as well as the axial tensors  $K^{Va}$  and  $K^{aV}$  need not fulfill any conditions of symmetry because they couple fluxes and forces which possess no symmetry properties. Also, it is trivial that the scalar coefficient and all vector coefficients of polar or axial character are not subjected to any symmetries.

#### b Forced Conditions Of Symmetry

In addition to the fundamental conditions of symmetry we must consider that in many applications it is meaningful to make use of further symmetry conditions which are not included in Eqs. (3-9) to (3-12) and with the help of which the complexity of the exchange tensors can be considerably reduced. Such symmetries can be required as additional "forced conditions" in accordance with natural symmetries of the

physical structure of the fluid or the flow.

Let us, first, briefly discuss the characteristics of transformation which are significant for the exchange coefficient tensors of Eqs. (3-5) to (3-8), assuming arbitrary conditions of symmetry which are included in the group of orthogonal transformations. Using the theories of de Groot and Mazur (1969) and/or Smith and Rivlin (1957) we find that an arbitrary exchange tensor is transformed with respect to a transformation  $A$ , where  $|A| = \pm 1$ , in component or symbolic form by the relations

$$K'_{i_1 i_2 \dots i_n} = \quad (3-13)$$

$$|A|^\mu \sum_{j_1 j_2 \dots j_n} A_{i_1 j_1} A_{i_2 j_2} \dots A_{i_n j_n} K_{j_1 j_2 \dots j_n}$$

or  $K' = |A|^\mu A^n(\cdot) K$

where  $i_1, i_2, \dots, i_n$  and  $j_1, j_2, \dots, j_n$  are indices for the cartesian components of the tensors  $K$  and  $A$  and where  $\mu = 0$  for polar tensors  $K$  and  $\mu = 1$  for axial tensors  $K$ .

If the exchange processes do not possess further properties of symmetry, except the index symmetries (3-9) to (3-12), it follows that the transformed  $K'$ -tensors of order 1 and higher (Eq. 3-13) are generally different from the non-transformed ones. But if there is an additional condition of symmetry which belongs to the transformation  $A$  the exchange tensors remain invariant under this transformation, i. e.

$$K' = K, \text{ and } |A|^\mu A^n(\cdot) K = K \quad (3-14)$$

Equation (3-14) allows to determine the structure of the constitutive coefficient scheme if, the matrix  $A$ , which analytically specifies the symmetries of the system, is inserted into Eq. (3-14). In order to determine the coefficient structure with respect to the symmetry characteristics related to any spatial directions, such as it is done in this paper for the case of axisymmetry, the invariant condition (Eq. 3-14) can be more suitably expressed in terms of scalar invariants. Finally, from the

procedure discussed in the following sections the axisymmetric form of each coefficient of the scheme of table 1 can be obtained without expressing the transformation  $A$  in form of an explicit matrix.

#### 4 Axisymmetric Coefficients and Transports

Instead of the various tensors of the coefficient scheme of table 1 we shall consider the cartesian tensor  $K_{ijkl\dots}$  of arbitrary order and of polar or axial character. Its spatial structure is assumed to be an explicit function of a definite direction  $\vec{\lambda}$  describing the axis of symmetry. Let  $\vec{a}, \vec{b}, \vec{c}, \vec{d}$  etc. denote arbitrary unit vectors (see Fig. 1); the fundamental scalar form of any coefficient tensor of arbitrary order is then expressible as the scalar product

$$K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d} \dots) = \sum_{i,j,k,l,\dots} K_{ijkl\dots}(\vec{\lambda}) a_i b_j c_k d_l \dots \quad (4-1)$$

in terms of the direction cosines  $a_i, b_i, c_i, d_i$ , etc. of the polar vectors  $\vec{a}, \vec{b}, \vec{c}, \vec{d}$  etc.

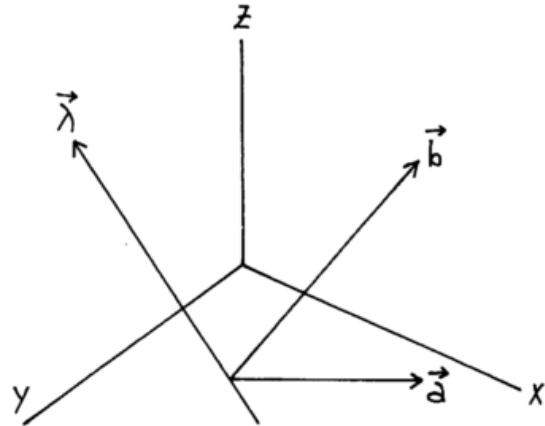


Figure 1: A symbolic representation of the configuration of the unit vectors

If we now consider any arbitrary rotation or reflexion of the axes of reference it must be required that the axis of symmetry remains unaltered, i. e.  $\vec{\lambda}' = \vec{\lambda}$ . With the use Eq. (4-1) the invariant condition (3-14) is then expressible, in accordance with the assumption of

axisymmetry, by

$$K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d} \dots) = K(\vec{\lambda}; \vec{a}', \vec{b}', \vec{c}', \vec{d}' \dots) \quad (4-2)$$

Also applying Eq. (3-14), together with Eqs. (4-1,2), it must additionally be considered with respect to the tensor invariance, in accordance with a fundamental theorem of the invariant theory, that the scalar forms of axial tensors  $K_{ijkl\dots}$  are scalar functions of the (polar) vectors  $\vec{\lambda}, \vec{a}, \vec{b}, \vec{c}, \vec{d} \dots$  which change sign if this vector configuration is subjected to a reflexion in the origin; thus we have, for axial tensors under reflexions, the invariance condition

$$K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d} \dots) = -K(\vec{\lambda}; \vec{a}', \vec{b}', \vec{c}', \vec{d}' \dots) \quad (4-2a)$$

since  $\mu=1, |A|=-1$  and  $|A|^\mu=-1$

Both equations (4-2) and (4-2a), however, state analytically that, at any arbitrary movement out of the full rotation group, by means of which the vector configuration  $\vec{\lambda}, \vec{a}, \vec{b}, \vec{c}, \vec{d}$  etc. is transformed into  $\vec{\lambda}, \vec{a}', \vec{b}', \vec{c}', \vec{d}'$  etc., the tensor  $K_{ijkl\dots}$  remains unchanged

$$K_{ijkl\dots}(\vec{\lambda}) \Big|_{\vec{a}, \vec{b}, \vec{c}, \vec{d} \dots} = K_{ijkl\dots}(\vec{\lambda}) \Big|_{\vec{a}', \vec{b}', \vec{c}', \vec{d}' \dots} \quad (4-3)$$

Thus, as clearly pointed out in the procedures used by Robertson (1940), Batchelor (1946) and Chandrasekhar (1950)<sup>+</sup>, the general problem, formulated in Eq. (3-14), can be restated for

the axisymmetric case, in terms of Eqs.(4-2,2a) which means that the most general function  $K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d} \dots)$  must be determined which is a scalar invariant under an arbitrary proper or improper rotation of the vector configuration formed by  $\vec{\lambda}, \vec{a}, \vec{b}, \vec{c}, \vec{d}$ , etc. where  $\vec{\lambda}$  will not be changed. Then, because of Eq. (4-3), the axisymmetric structure of the tensor  $K_{ijkl\dots}(\vec{\lambda})$  is determined as well, (see also Panchev, 1971).

Firstly, we recognize from Eq. (4-1) that the scalar form of the tensor  $K_{ijkl\dots}$  is to be determined as a sum of homogeneous and linear functions of the configuration  $a_i b_j c_k d_l$  etc. This problem is readily solved since it follows from the theory of invariants (to which Robertson (1940) first directed attention) that such a scalar function of any number of vectors  $\vec{\lambda}, \vec{a}, \vec{b}, \vec{c}, \vec{d} \dots$  must be expressible only in terms of the fundamental invariants satisfying the same conditions, namely by

$$\left. \begin{aligned} (a) \quad & \vec{\lambda} \cdot \vec{\lambda}, \vec{\lambda} \cdot \vec{a}, \vec{\lambda} \cdot \vec{b}, \dots \\ & \vec{a} \cdot \vec{a}, \vec{a} \cdot \vec{b}, \dots \vec{d} \cdot \vec{d} \text{ etc.} \\ (b) \quad & \vec{\lambda} \cdot (\vec{a} \times \vec{b}), \vec{\lambda} \cdot (\vec{a} \times \vec{c}), \\ & \vec{\lambda} \cdot (\vec{b} \times \vec{c}), \vec{\lambda} \cdot (\vec{c} \times \vec{d}) \text{ etc.} \end{aligned} \right\} (4-4)$$

With respect to the scalar invariants of Eq. (4-4) must be noted that, according to the theory of invariance, tensors of polar character and their corresponding forms depend only on the scalar invariants of type (a) while axial tensors and their skew forms must only be expressible as sums of products of an odd number of determinants such as  $\vec{\lambda} \cdot (\vec{a} \times \vec{b})$  of the invariant type (b) (Chandrasekhar, 1950). Consequently, choosing the combinations of the available invariants of type (a) and (b), which must be required in conformance with relation (4-1), the most general axisymmetric forms and tensors may be evaluated. In table 2 we summarized, as an essential result of these calculations specified completely in Appendix B, all the needed invariant combinations of the

<sup>+</sup> See, also, Smith and Rivlin (1957) and Ghosh (1968) of whom the latter applied the invariance theory in order to examine the behaviour of the stress tensor  $\sigma_{ij}$  in an axisymmetric stream including fluid inhomogeneities.

fundamental scalar products or determinants which are correlated to each exchange coeffi-

cient of the scheme of table 1.

| order | Polar Coefficients                   |  |  | Axial Coefficients           |   |  |
|-------|--------------------------------------|--|--|------------------------------|---|--|
|       | tensors                              | invariant form   | scalar invariants  | tensors                      | invariant form                                | deter-<br>minants                              |
| 0     | $K^{ss}$                             | $K^{ss}$   | $K^{ss}$   | —                            | —   | —  |
| 1     | $\vec{K}^{sv}, \vec{K}^{vs}$         | $K(\vec{\lambda}; \vec{a})$                            | $\vec{\lambda} \cdot \vec{a}$  | $\vec{K}^{sa}, \vec{K}^{as}$ | $K(\vec{\lambda}; \vec{a})$                   | —  |
| 2     | $K^{vv}, K^{aa}$<br>$K^{st}, K^{ts}$ | $K(\vec{\lambda}; \vec{a}, \vec{b})$                   | $\vec{a} \cdot \vec{b}, (\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})$  | $K^{va}, K^{av}$             | $K(\vec{\lambda}; \vec{a}, \vec{b})$          | $\vec{\lambda} \cdot (\vec{a} \times \vec{b})$ |
| 3     | $K^{vt}, K^{tv}$                     | $K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c})$          | $(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c}),$<br>$(\vec{\lambda} \cdot \vec{a})(\vec{b} \cdot \vec{c}),$<br>$(\vec{\lambda} \cdot \vec{b})(\vec{a} \cdot \vec{c}), (\vec{\lambda} \cdot \vec{c})(\vec{a} \cdot \vec{b})$  | $K^{at}$<br>$K^{ta}$         | $K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c})$ | $\vec{a} \cdot (\vec{b} \times \vec{c})$       |
| 4     | $K^{tt}$                             | $K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d})$ | $(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c})(\vec{\lambda} \cdot \vec{d})$<br>$(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{c} \cdot \vec{d})$<br>$(\vec{\lambda} \cdot \vec{c})(\vec{\lambda} \cdot \vec{d})(\vec{a} \cdot \vec{b})$<br>$(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{c})(\vec{b} \cdot \vec{d})$<br>$(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c})(\vec{a} \cdot \vec{d})$<br>$(\vec{\lambda} \cdot \vec{d})(\vec{\lambda} \cdot \vec{a})(\vec{b} \cdot \vec{c})$<br>$(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{d})(\vec{a} \cdot \vec{c})$<br>$(\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$<br>$(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d})$<br>$(\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d})$ | —                            | —   | —  |

Table 2: Scalar invariants of the axisymmetric coefficient tensors

Note, that the analytical structure of the fundamental axisymmetric invariants, characterized in table 2 by the forms such as  $K(\vec{\lambda}; \vec{a}, \vec{b})$  result by inner multiplications according to Eq. (4-1). In addition, we must note in table 2 that, in consistency with Eqs. (4-2) and (4-2a), the forms of axial tensors transform as ordinary polar tensors under proper rotations but they take the opposite sign on a reflexion in the origin.

On the basis of those invariant expressions presented in table 2, the axisymmetric structure of the exchange coefficients has been calculated in Appendix B. The results of these calculations shall now be summed discussed for

each coefficient tensor type:

i Coefficient-Tensors Of 1. Order (Vectors)  
(See Also Fig. 2)

Polar structure:  $K_i = \mathcal{L} \lambda_i$  or  $\vec{K} = \mathcal{L} \vec{\lambda}$   
where  $\mathcal{L}$  = scalar constant. This means that the components of the polar exchange tensors of first order are determined by one single scalar each:

$$\vec{K}^{sv} = \mathcal{L}^{sv} \vec{\lambda}, \vec{K}^{vs} = \mathcal{L}^{vs} \vec{\lambda},$$

$$\text{or } K_i^{sv} = \mathcal{L}^{sv} \lambda_i, K_i^{vs} = \mathcal{L}^{vs} \lambda_i \quad (4-5)$$

$$(i = 1, 2, 3)$$

where the  $\lambda_i$  are the direction cosines of  $\vec{\lambda}$ . The scalars  $\mathcal{L}^{SV}, \mathcal{L}^{VS}$  are, in principle, different from each other. Also, it is clear that a vector which is invariant under the full rotation group can be described by its value in the invariant direction. Hence, inserting the coefficients  $\vec{K}^{SV}$  and  $\vec{K}^{VS}$  in their axisymmetric formulations into the corresponding expressions of Eqs. (3-5) and (3-6), we find

$$\vec{K}^{SV} \cdot \nabla T = \mathcal{L}^{SV} \vec{\lambda} \cdot \nabla T, \quad \vec{K}^{VS} \cdot \nabla \vec{u} = \mathcal{L}^{VS} \vec{\lambda} \cdot \nabla \vec{u} \quad (4-6)$$

which means that the cross-effects between the scalar friction flux  $\mathcal{f}$  and the heat flux  $\vec{I}_h$  are expressible by two scalar coefficients  $\mathcal{L}^{SV}$  and  $\mathcal{L}^{VS}$ , only. Finally if the axis of symmetry is considered to be equal to the unit

With respect to the axial vectors  $\vec{K}^{sa}$  and  $\vec{K}^{as}$  (see table 1) we must recognize that the corresponding axisymmetric form  $\sum K_i(\vec{\lambda}) a_i$  vanishes identically, since with the two vectors  $\vec{\lambda}$  and  $\vec{a}$  no determinant of the invariant type (b) (see Eq. 4-4) can be constructed. Hence,

$$K_i^{sa} = 0, \quad K_i^{as} = 0 \quad (4-8)$$

indicating that the fluxes of volume and rotational friction do not interfere. Thus, the corresponding axisymmetric cross-effects between bulk-viscosity and rotational friction transport processes must disappear in Eqs. (3-5) and (3-7):

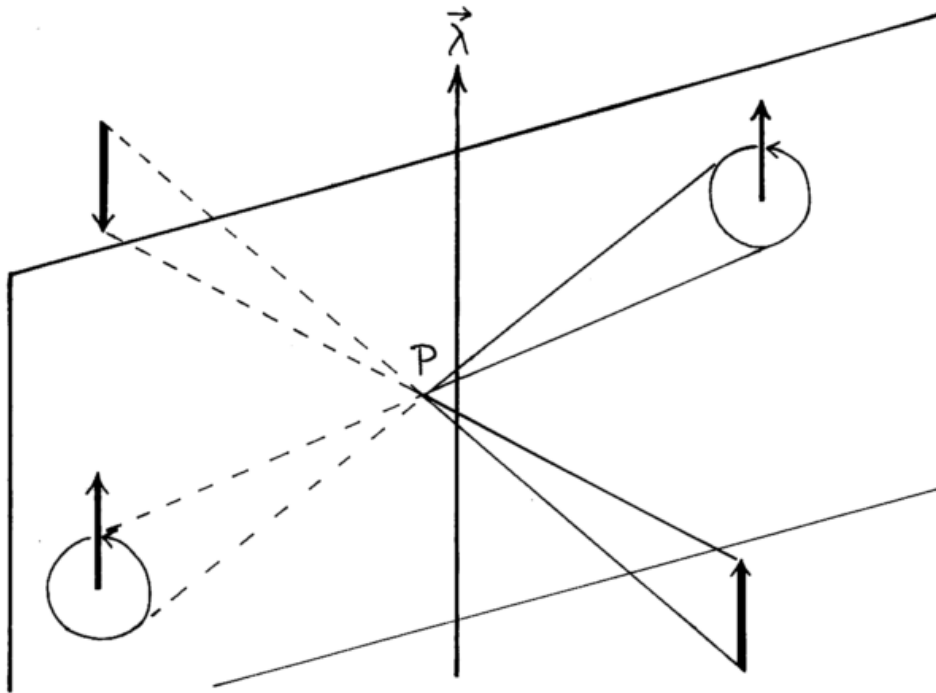


Figure 2: Inversion of a polar and axial vector with respect to a plane directed through the  $\vec{\lambda}$ -axis

vector of the vertical direction we obtain, introducing  $\vec{\lambda} = \vec{e}_z$  instead of Eq. (4-6), the formulas

$$K^{SV} \cdot \nabla T = \mathcal{L}^{SV} \frac{\partial T}{\partial z}, \quad K^{VS} \cdot \nabla \vec{u} = (\mathcal{L}^{VS} \nabla \cdot \vec{u}) \vec{e}_z \quad (4-7)$$

Equations (4-6,7) denote that cross-exchange processes between the fluxes of heat and volume viscosity are under axisymmetric transport conditions merely possible along the axis of symmetry.

$$\begin{aligned} \vec{K}^{sa} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) &= 0 \\ \vec{K}^{as} \cdot \nabla \cdot \vec{u} &= 0 \end{aligned} \quad (4-9)$$

ii Coefficient-Tensors Of 2nd Order

Polar structure:  $K_{ij} = A \lambda_i \lambda_j + B \delta_{ij}$   
or  $K = A \vec{\lambda} \vec{\lambda} + B \delta$

Axisymmetric coefficients of 2nd order are thus expressible by the scalars  $A$  and  $B$ . Furthermore the conditions of symmetry for different

indices,  $i \neq j$  are, in accordance with Eq.(3-9) ipso facto fulfilled. Requiring in addition the tensor trace to be zero  $K_{ij}$  is described by only one coefficient:

$$K_{ij} = A(\lambda_i \lambda_j - \frac{1}{3} \delta_{ij}) \text{ or } K = A(\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta)$$

From the formulas (3-6,7) it is seen that the coefficients which relate vectorial fluxes and forces are not subjected to zero trace so that we obtain

$$\left. \begin{aligned} K_{ij}^{vv} &= A^{vv} \lambda_i \lambda_j + B^{vv} \delta_{ij}, \\ \text{or } K^{vv} &= A^{vv} \vec{\lambda} \vec{\lambda} + B^{vv} \delta \\ K_{ij}^{da} &= A^{da} \lambda_i \lambda_j + B^{da} \delta_{ij}, \\ \text{or } K^{da} &= A^{da} \vec{\lambda} \vec{\lambda} + B^{da} \delta \end{aligned} \right\} (4-10)$$

as well as for the corresponding partial fluxes of heat conduction and of friction exchange due to rotational effects:

$$\begin{aligned} K^{vv} \cdot \nabla T &= \\ &= (A^{vv} \vec{\lambda} \vec{\lambda} + B^{vv} \delta) \cdot \nabla T \\ K^{da} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) &= \\ &= (A^{da} \vec{\lambda} \vec{\lambda} + B^{da} \delta) \cdot (\nabla \times \vec{u} - 2\vec{\omega}) \end{aligned} \quad (4-11)$$

Equation (4-11) documents clearly the influence of the axis of symmetry on vectorial exchange phenomena. Beside familiar isotropic exchange processes of heat and rotational friction there are additional terms, due to the invariant direction  $\vec{\lambda}$ , coupled with the constitutive coefficients  $A^{da}$  and  $A^{vv}$ .

Coefficients which connect scalar (or dyadic) fluxes with dyadic (or scalar) forces have no trace according to Eq. (3-9). With their canonical form (see Eq. B-5) we then proceed to

$$\left. \begin{aligned} K_{ij}^{st} &= A^{st}(\lambda_i \lambda_j - \frac{1}{3} \delta_{ij}), \text{ or} \\ K^{st} &= A^{st}(\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta) \\ K_{ij}^{ts} &= A^{ts}(\lambda_i \lambda_j - \frac{1}{3} \delta_{ij}), \text{ or} \\ K^{ts} &= A^{ts}(\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta) \end{aligned} \right\} (4-12)$$

Inserting Eq. (4-12) into the corresponding expressions for the frictional cross-processes of Eqs. (3-5) and (3-8) we obtain

$$\left. \begin{aligned} K^{st} : (\nabla \vec{u})^{0,s} &= \\ &= A^{st}(\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta) : (\nabla \vec{u})^{0,s} \\ K^{ts} \nabla \cdot \vec{u} &= \\ &= A^{ts}(\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta) \nabla \cdot \vec{u} \end{aligned} \right\} (4-13)$$

so that only 2 values  $A^{st}$  and  $A^{ts}$  are needed to describe the 18 cross-coefficients in Eq.(4-13).

If we are interested in knowing the partial fluxes of Eqs. (4-11) and (4-13) in which the coefficients are vertically axisymmetric the invariant direction  $\vec{\lambda}$  is to be replaced by  $\vec{e}_z$ , being the unit vector of the vertical direction. Hence, the vertically axisymmetric structure of Eqs. (4-11,13) may be formulated as:

$$\begin{aligned} K^{vv} \cdot \nabla T &= \\ &= \left\{ (A^{vv} + B^{vv}) \vec{e}_z \vec{e}_z + B^{vv} \delta_H \right\} \cdot \nabla T \end{aligned} \quad (4-14)$$

$$\begin{aligned} K^{da} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) &= \\ &= \left\{ (A^{da} + B^{da}) \vec{e}_z \vec{e}_z + B^{da} \delta_H \right\} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) \end{aligned}$$

and

$$\begin{aligned} K^{st} : (\nabla \vec{u})^{0,s} &= \\ &= \frac{A^{st}}{3} (2 \vec{e}_z \vec{e}_z - \delta_H) : (\nabla \vec{u})^{0,s} \end{aligned} \quad (4-15)$$

$$\begin{aligned} K^{ts} \nabla \cdot \vec{u} &= \\ &= \frac{A^{ts}}{3} (2 \vec{e}_z \vec{e}_z - \delta_H) \nabla \cdot \vec{u} \end{aligned}$$

where  $\delta_H = \vec{e}_x \vec{e}_x + \vec{e}_y \vec{e}_y$ , being the horizontal unit tensor.

Axial structure:  $K_{ij} = \rho \sum_k \epsilon_{ijk} \lambda_k$ , or

$$K = \rho \{ \lambda_x (\vec{e}_y \vec{e}_z - \vec{e}_z \vec{e}_y) + \lambda_y (\vec{e}_z \vec{e}_x - \vec{e}_x \vec{e}_z) + \lambda_z (\vec{e}_x \vec{e}_y - \vec{e}_y \vec{e}_x) \}$$

which can alternatively be written as a vector

$$\frac{1}{2} K_x = \rho (\lambda_x, \lambda_y, \lambda_z) = \rho \vec{\lambda}. \text{ Correspondingly}$$

we obtain for the coefficients  $K^{va}$  and  $K^{av}$  (see table 1) the axisymmetric forms represented in vector notation

$$\frac{1}{2} K_x^{va} = \rho^{va} \vec{\lambda} \quad \text{and} \quad \frac{1}{2} K_x^{av} = \rho^{av} \vec{\lambda} \quad (4-16)$$

Then results from straight-forward calculation that the irreversible cross-effects between the heat flux and the axial component of the friction flux (see Eq. 3-6, 7) can be written as

$$\begin{aligned} K^{va} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) &= \\ &= -\rho^{va} \vec{\lambda} \times (\nabla \times \vec{u} - 2\vec{\omega}) \end{aligned} \quad (4-17)$$

$$K^{av} \cdot \nabla T = -\rho^{av} \vec{\lambda} \times \nabla T$$

When  $\vec{\lambda}$  is assumed to coincide with the unit vector of the vertical direction, i. e.  $\vec{\lambda} = (0, 0, 1) = \vec{e}_z$ , Eq. (4-17) resolves as follows

$$\begin{aligned} K^{va} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) &= \\ &= -\rho^{va} \left[ \nabla w - \frac{\partial \vec{u}}{\partial z} - 2(\omega_x \vec{e}_x - \omega_y \vec{e}_y) \right] \end{aligned} \quad (4-18)$$

$$\begin{aligned} K^{av} \cdot \nabla T &= \\ &= -\rho^{av} \left[ \frac{\partial T}{\partial x} \vec{e}_x - \frac{\partial T}{\partial y} \vec{e}_y \right] \end{aligned}$$

We wish to stress that all relations formulated in Eq. (4-13,15,17, and 18) document instructively a characteristic property of axisymmetric exchange: the interference between the momentum flux components due to bulk and shear viscosity as well as the interference between the momentum flux component due to rotational friction and the conductive heat flux.

### iii Coefficient-Tensors Of Third Order

As calculated in Eqs. (B-11) and (B-13) all axisymmetric third order coefficients in the scheme of table 1 must vanish since their natural symmetries, formulated in Eq. (3-11), must be taken

into account. Note also that the corresponding flux contributions which are related to these coefficients in Eqs. (3-6) to (3-8) thus disappear:

$$\begin{aligned} K^{vt} : (\nabla \vec{u})^{0, S} &= 0 \\ K^{tv} \cdot \nabla T &= 0 \\ K^{at} : (\nabla \vec{u})^{0, S} &= 0 \\ K^{ta} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) &= 0 \end{aligned} \quad (4-19)$$

This expresses that there are no axisymmetric cross-exchange processes neither between the constitutive equations of the heat flux  $\vec{I}_h$  and the friction fluxes  $\vec{I}^a$  and  $\vec{I}^{0, S}$  nor between these friction fluxes themselves.

### iiii Coefficient-Tensors Of Fourth Order

There is in table 1 only one coefficient of fourth order. Because of its polar character, its general axisymmetric structure is expressible according to Eq. (B-14) in terms of 10 scalars as follows

$$\begin{aligned} K_{ijke} &= A \lambda_i \lambda_j \lambda_k \lambda_e + \\ &+ B \lambda_i \lambda_j \delta_{ke} + C \lambda_k \lambda_e \delta_{ij} + \\ &+ D \lambda_i \lambda_k \delta_{je} + E \lambda_j \lambda_k \delta_{ie} + \\ &+ F \lambda_i \lambda_e \delta_{jk} + G \lambda_j \lambda_e \delta_{ik} + \\ &+ H \delta_{ij} \delta_{ke} + I \delta_{ik} \delta_{je} + K \delta_{ie} \delta_{jk} \end{aligned} \quad (4-20)$$

On account of the symmetry conditions (Eq. 3-12) only the three parameters B, D, H are left as independents (see Eq. B-23) instead of which the new parameters  $B = (K_H - K_{HH})/2$ ,  $D = (K_{HV} - K_H)/2$ ,  $H = -K_{HH}/3 - (K_H - K_{HH})/2$  shall be introduced alternatively. In terms of the new coefficients  $K_H$ ,  $K_{HH}$  and  $K_{HV}$  we then obtain for the tensor structure of the

fourth order tensor:

$$\begin{aligned}
 K_{ijkl} = & \quad (4-21) \\
 = & \frac{K_H}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{K_{HH}}{3} \delta_{ij} \delta_{kl} \\
 + & \frac{K_H - K_{HH}}{2} (\lambda_i \lambda_j \delta_{kl} + \lambda_k \lambda_l \delta_{ij} - \delta_{ij} \delta_{kl}) \\
 + & \frac{K_{HV} - K_H}{2} (\lambda_i \lambda_k \delta_{jl} + \lambda_j \lambda_k \delta_{il} + \\
 & \quad + \lambda_i \lambda_l \delta_{jk} + \lambda_j \lambda_l \delta_{ik}) + \\
 + & \frac{1}{2} (K_H + 3K_{HH} - 4K_{HV}) \lambda_i \lambda_j \lambda_k \lambda_l
 \end{aligned}$$

in accordance with Eq.(B-25) This version (Eq.4-21) can even be simplified in considering that  $K_{ijkl}^{tt}$  will exclusively be used as factor of the deformation tensor in the double-scalar product  $K^{tt} : (\nabla \vec{u})^{0,s}$ , being part of the friction deviator  $\mathbf{I}^{0,s}$  of Eq. (3-8). Note that the multiplication with the strain rate tensor must be carried out with respect to the last pair of indices of  $K_{ijkl}^{tt}$  employing it in the form of Eq.(4-21). As a result we obtain from this procedure that the 9 components of that expression can be deduced from

$$\begin{aligned}
 \sum_{k,l} K_{ijkl} \left( \frac{\partial u_k}{\partial x_l} \right)^{0,s} = & \quad (4-22) \\
 = & \frac{K_H}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \nabla \cdot \vec{u} \delta_{ij} \right) + \sum_{k,l} \left\{ \right. \\
 & \frac{1}{4} \left[ (K_H - K_{HH}) \delta_{ij} \lambda_k \lambda_l + (K_{HV} - K_H) * \right. \\
 & * (\lambda_i \lambda_k \delta_{jl} + \lambda_j \lambda_k \delta_{il} + \lambda_i \lambda_l \delta_{jk} + \lambda_j \lambda_l \delta_{ik}) \\
 & + (K_H + 3K_{HH} - 4K_{HV}) \lambda_i \lambda_j \lambda_k \lambda_l \left. \right] * \\
 & \left. * \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} - \frac{2}{3} \nabla \cdot \vec{u} \delta_{kl} \right) \right\}
 \end{aligned}$$

denoting that the  $\delta_{kl}$ -terms of the coefficient presented in Eq. (4-21) do not contribute

in any way to the friction flux. Hence, is sufficiently expressible without these terms by means of the approach:

$$\begin{aligned}
 K_{ijkl}^{tt} = & \quad (4-23) \\
 & \frac{K_H}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{K_H - K_{HH}}{2} \lambda_k \lambda_l \delta_{ij} \\
 + & \frac{K_{HV} - K_H}{2} \left[ \lambda_i \lambda_k \delta_{jl} + \lambda_j \lambda_k \delta_{il} + \right. \\
 & \quad \left. + \lambda_i \lambda_l \delta_{jk} + \lambda_j \lambda_l \delta_{ik} \right] + \\
 + & \frac{1}{2} (K_H + 3K_{HH} - 4K_{HV}) \lambda_i \lambda_j \lambda_k \lambda_l
 \end{aligned}$$

Note in this connection, that we will confine ourselves to formulate the axisymmetric structure of the fourth-ranked tensor  $K^{tt}$  only in the index version (Eq.4-23), since the tensorial form is not very instructive.

In connection with Eq. (3-8) the three parameters  $K_H$ ,  $K_{HH}$  and  $K_{HV}$  have the meaning of shear-viscous coefficients. If, as it is usually done in many simulation models of atmospheric and oceanic circulation, the processes of bulk-viscosity and rotational friction due, to Eqs. (3-5) and (3-7) are ignored, the momentum transfer is completely represented by the shear viscous flux  $\mathbf{I}^{0,s}$  which is then sufficiently described in the form of Eq. (4-22) including the frictional coefficients  $K_H$ ,  $K_{HH}$  and  $K_{HV}$ . Employing, instead of Eq. (4-22), the tensorial formulation we proceed to the equivalent relation:

$$\begin{aligned}
 K^{tt} : (\nabla \vec{u})^{0,s} = & \quad (4-24) \\
 = & \frac{K_H}{2} (\nabla \vec{u} + \vec{u} \nabla - \frac{2}{3} \nabla \cdot \vec{u} \delta) + \\
 + & \frac{1}{4} \left[ (K_H - K_{HH}) \delta \vec{\lambda} \vec{\lambda} + (K_{HV} - K_H) * \right. \\
 & * (\vec{\lambda} \vec{e} \vec{\lambda} \vec{e} + \vec{e} \vec{\lambda} \vec{\lambda} \vec{e} + \vec{\lambda} \vec{e} \vec{e} \vec{\lambda} + \vec{e} \vec{\lambda} \vec{e} \vec{\lambda}) + \\
 & \left. + (K_H + 3K_{HH} - 4K_{HV}) \vec{\lambda} \vec{\lambda} \vec{\lambda} \vec{\lambda} \right] : \\
 & (\nabla \vec{u} + \vec{u} \nabla - \frac{2}{3} \nabla \cdot \vec{u} \delta)
 \end{aligned}$$

where  $\vec{e}$  is a unit vector denoting symbolically the direction of the cartesian axes of coordinates. The last terms of Eq. (4-24) where the tensor of strain must be double-scalarly multiplied with the exchange coefficients could still be reformulated in a more instructive manner if the axis of symmetry will be explicitly specified. In order to do this we will write down Eq. (4-24) for that case in which the unit vector of the axis of symmetry coincides with the unit vector of the vertical direction of a particular x, y, z-cartesian system of coordinates. Inserting  $\vec{\lambda} = (0, 0, 1)$  and arranging Eq. (4-24) in terms of  $K_{HH}$ ,  $K_{HH}$  and  $K_{HV}$ , we obtain

$$\begin{aligned}
 & K^{tt} : (\nabla \vec{u})^{0,1,5} = \\
 & = \frac{K_H}{2} (\nabla_H \vec{u}_H + \vec{u}_H \nabla_H - \nabla \cdot \vec{u}_H \delta_H) + \\
 & + \frac{K_{HH}}{2} \left( \frac{\nabla \cdot \vec{u}}{3} - \frac{\partial \omega}{\partial z} \right) (\delta_H - 2 \vec{e}_z \vec{e}_z) \quad (4-25) \\
 & + \frac{K_{HV}}{2} \left[ \vec{e}_z \left( \frac{\partial \vec{u}_H}{\partial z} + \nabla_H \omega \right) + \right. \\
 & \quad \left. + \left( \frac{\partial \vec{u}_H}{\partial z} + \nabla_H \omega \right) \vec{e}_z \right]
 \end{aligned}$$

where  $\delta_H = \vec{e}_x \vec{e}_x + \vec{e}_y \vec{e}_y$ ,  $\nabla_H = \vec{e}_x \partial/\partial x + \vec{e}_y \partial/\partial y$ ,  $\vec{u}_H = \vec{e}_x u + \vec{e}_y v$ . We stress that the form of the shear frictional exchange expressions specified in Eq. (4-25) is equivalent to that recently found by Herbert (1975) on the basis of the Curie-theorem applied to isotropic cartesian directions.

## 5 Applications to Isotropic Systems

### a Governing Equations

It is well-known that isotropy represents the strongest restriction within the set of spatial symmetry conditions. Expressed in the language of the theory of invariants an isotropic tensor of arbitrary order and polar character is invariant under all (proper) rotations and reflections. Considering axial tensors - or skew

tensors as they are sometimes called - it follows from Eq. (3-14) that they transform as ordinary tensors under proper rotations but they take the opposite sign to true (polar) tensors on inversions. In comparison to axisymmetric coefficients, formulated in section 4 above, the single difference characterizing isotropic coefficients is their independence on any invariant distinguished direction. From Eqs. (4-2) and (4-2a) it is therefore clear that the invariant condition for the scalar forms of isotropic exchange coefficients of order  $n > 0$  then reads

$$K(\vec{a}, \vec{b}, \vec{c}, \vec{d} \dots) = \begin{cases} K(\vec{a}', \vec{b}', \vec{c}', \vec{d}' \dots) \\ \text{for proper rotations} \\ -K(\vec{a}', \vec{b}', \vec{c}', \vec{d}' \dots) \quad (5-1) \\ \text{for inversions} \end{cases}$$

Consequently, the isotropic structure of the exchange coefficients should be directly concluded from their corresponding axisymmetric forms when the condition  $\vec{\lambda} = (0, 0, 0)$  is applied to these expressions.

By virtue of this hypothesis we find from section 4 the non-vanishing isotropic coefficients

$$K_{ij}^{vv} = K_h \delta_{ij}, \text{ or } K^{vv} = K_h \delta \quad (5-2)$$

$$K_{ij}^{aa} = K_r \delta_{ij}, \text{ or } K^{aa} = K_r \delta \quad (5-3)$$

$$K_{ijkl}^{tt} = K_m \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \quad (5-4)$$

Therein the more suitable notations  $K_h$  and  $K_r$  have been introduced instead of the coefficients  $B^{vv}$  and  $B^{aa}$ , where  $K_h$  denotes isotropic energy transport by the heat flux and  $K_r$  denotes isotropic momentum transport by the rotational friction stress. The coefficient  $K_m$  has been utilized instead of  $K_H$  denoting isotropic momentum exchange by the shear-friction stress. This coefficient remains left in Eq. (4-21) as the single coefficient if one considers that in addition to  $\vec{\lambda} = (0, 0, 0)$  also the isotropic condition  $K_{HH} = K_H$  must be valid for satisfying

the trace-conditions due to Eq. (3-12). Furthermore follows obviously from the isotropic version of Eq. (4-23) that Eq. (5-4) can also be written without the  $\delta_{k\ell}$ -term since  $K_{ijk\ell}^{tt}$  is for the present purposes a well-defined physical quantity, exclusively applied as factor of the deformation deviator  $(\partial u_k / \partial x_\ell)^{0,s}$ . Hence, as isotropic factor of the deformation deviator we have

$$K_{ijk\ell}^{tt} = K_m (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \quad (5-5)$$

Note, that in the scheme of coefficients of table 1 all those coefficients which connect fluxes and forces of the same tensor order and character are, (Eqs. (5-2) to (5-5)), expressible by one scalar each. But those coefficients which connect fluxes and forces of different order or character will disappear, which can easily be shown as follows:

Equations (4-5), (4-7) and (4-15) together with  $\vec{\lambda} = 0$  yield immediately

$$\left. \begin{aligned} K_i^{sv} = 0, \quad K_i^{vs} = 0 \\ K_i^{sa} = 0, \quad K_i^{as} = 0 \\ (i=1,2,3) \end{aligned} \right\} \quad (5-6)$$

$$K_{ij}^{va} = 0, \quad K_{ij}^{av} = 0 \quad (i,j)=1,2,3 \quad (5-7)$$

From the axisymmetric polar coefficients of 2nd order we firstly obtain for vanishing  $\vec{\lambda}$  that  $K_{ij}^{st} = -\frac{1}{3} A^{st} \delta_{ij}$  and  $K_{ij}^{ts} = -\frac{1}{3} A^{ts} \delta_{ij}$ . Also, to satisfy that the coefficient-traces are zero, in accordance with Eq. (3-9)  $A^{st} = 0$  and  $A^{ts} = 0$  as well as

$$K_{ij}^{st} = 0, \quad K_{ij}^{ts} = 0 \quad (i,j)=1,2,3 \quad (5-8)$$

must be valid. Moreover, all isotropic coefficients of third order are zero, which is obvious from their vanishing axisymmetric forms (see Section 4, c).

We find with all the results obtained for isotropic coefficients that instead of Eqs. (3-5) to (3-8) the linear constitutive equations

$$\begin{aligned} \vec{f} &= -K_v \nabla \cdot \vec{u}, \quad \vec{I}_h = -K_h \nabla T \\ \vec{I}^a &= -K_r (\nabla \times \vec{u} - 2\vec{\omega}) \end{aligned} \quad (5-9)$$

$$\underline{I}^{0,s} = -\frac{K_m}{2} (\nabla \vec{u} + \vec{u} \nabla - \frac{2}{3} \nabla \cdot \vec{u} \delta)$$

have to be used where the four coefficients  $K_v$  (instead of  $K^{ss}$ ),  $K_h$ ,  $K_r$  and  $K_m$  are scalars with which the isotropic irreversible exchange of heat and friction is completely described. The practical meaning of the coefficients given in Eq. (5-9) is evident, particularly, if the actual forces coupled with them are taken into consideration. Thus the coefficient  $K_h$  denotes the pure heat conductivity while  $K_v$  denotes the volume,  $K_r$  the rotational and  $K_m$  the shear friction (or viscosity) coefficient, respectively, effected by the compressibility, the rotational and the shear behaviour of the fluid.

Equation (5-9) can also be expressed in such a way that for the case of isotropic systems all exchange coefficients occurring in linear constitutive equations of the irreversible fluxes are reduced to a scalar multiplied by the unit tensor. Hence, in analogy to  $K^{vv}$  and  $K^{aa}$  of Eqs. (5-2,3), the coefficient  $K^{tt}$  can be written as

$$K^{tt} \longrightarrow K^{tt} = K_m \delta \quad (5-10)$$

It should be remarked that if, more than one vectorial phenomenon exist, as in the presence of diffusive mass fluxes among the constituent substances of the system (e.g. air, water vapor, aerosol particles), we have cross-effects which couple the heat and the diffusive mass fluxes. However, these cross effects possess according to the theory outlined here, also scalar coefficients expressible in the form Eqs. (5-2,3 and 10). The tensor structure of the constitutive equations written down in Eq. (5-9) is then completely justified as well. In conclusion we can thus remark that we succeeded in deriving from axisymmetric systems with the assumption  $\vec{\lambda} = (0, 0, 0)$  the well-

known constitutive coefficients and equations which are usually written down for isotropic systems with the aid of the Curie-theorem (de Groot and Mazur, 1969). As it has been demonstrated by de Groot and Mazur this principle means in accordance with Eqs. (5-2) to (5-10) that only fluxes and forces of same tensorial order and character can be connected with isotropic coefficients, whereas conversely, fluxes and forces of different order or character do not interfere with one another.

b Discussion Of Eddy-Fluxes  
Of Averaged Turbulent Flow

For the description of eddying motion in the atmosphere and ocean the fluxes of turbulent friction, heat, and mass diffusion should be taken into account. By applying a statistical averaging operation to all terms of the original field differential equations of motion, energy and continuity, usually used in order to characterize fluid dynamic fields of turbulent flows, one obtains differential equations for the averaged field variables including as additional terms so-called eddy-fluxes. These fluxes represent statistical correlation products between the turbulent velocity fluctuations and field quantities denoting individual realisations of the averaged field variables such as mean momentum, enthalpy, temperature, and mass fractions of the constituent components (see for example van Mieghem, 1973).

By taking a density-weighted mean of the equations of fluid dynamics the flux of turbulent friction is then expressible in terms of the irreversible part of the Reynolds-stress (cf. Eq. 2-9):

$$\overline{I_{Re}} = \overline{\rho \vec{u}'' \vec{u}''} - \frac{2}{3} \rho E \delta \quad (5-11)$$

implying that rotational and bulk eddy-viscosity effects have been disregarded, i. e.

$$\vec{j} = 0 \quad \text{and} \quad \overline{\vec{I}^a} = 0 \quad (5-12)$$

With a procedure for the averaged momentum balance equation similar to that of sections 2

and 3 the relationships

$$\begin{aligned} \overline{I_{Re}}^{ij} &= \\ &= - \sum_{k,l} \frac{K_{ijkl}}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} - \frac{2}{3} \nabla \cdot \vec{u} \delta_{kl} \right) \quad (5-13) \\ &\quad (i,j) = 1,2,3 \end{aligned}$$

may be formulated as linear constitutive equations for the stress components. In Eq. (5-13) was considered that, because of the anisotropy of the length-scales in eddy-flow motions, the eddy-viscosities, denoted by the fourth-ranked tensor  $K_{ijkl}$ , must be assumed to be subjected to this anisotropy behaviour. From Eq. (5-11) it is clear that the eddy-friction flux is only subjected to its natural symmetries

$$\overline{I_{Re}}^{ij} = \overline{I_{Re}}^{ji} \quad \text{and} \quad \sum \overline{I_{Re}}^{ii} = 0 \quad (5-14)$$

Thus it will be reasonable for all practical applications to determine, under any given anisotropic restriction, the tensor structure of the coefficient  $K_{ijkl}$  instead of the flux density  $\overline{I_{Re}}^{ij}$  since the coefficient tensor has a more obvious geometrical meaning and its principal axes may sometimes be preassigned in consistency with realistic geometrical considerations. As a first rough approximation one, often, simply assumes analogously to molecular movements that the eddy-viscosity tensor  $K_{ijkl}$  may also be expressed as an isotropic tensor in terms of one scalar only; this yields in accordance with Eq. (5-10) and (5-9) the hypothetical formulation

$$K_{ijkl} \longrightarrow K_{ij} = k_m \delta_{ij} \quad (5-13a)$$

and

$$\overline{I_{Re}}^{ij} = - \frac{k_m}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \nabla \cdot \vec{u} \delta_{ij} \right)$$

which is close to that already provided by Boussinesq (1877, 1897). This hypothesis (Eq. 5-13a) is also applied and discussed in many modern works as for example by Harlow and Nakayama (1967) and by Lumley (1967). Note, however, that by applying the isotropy hypothesis to the eddy-flux of turbulent friction the constitutive equations require linear-homogeneous proportionality among stress and

strain rate components of the same index pair

$$\bar{I}_{Re}^{ij} \sim \left( \frac{\partial u_i}{\partial x_j} \right)^{0,5} \quad \text{all } (i,j) \quad (5-15)$$

Hence the principal axes of the stress and the strain rate tensors would be the same. However, it has been suspected that this is not true in real flows. This suspect has been confirmed by Monin and Yaglom (1971) and by Fiedler (1975) who used recent observations of Weseley (1974) for all elements of the Reynolds-stress within the turbulent atmospheric boundary layer, wherein the thermal stability varies within the range  $-1 \leq z/L \leq 1$ , with  $L$  being the Obukhov length. This confirmation implies that the direction of the principal axes of the stress and mean strain rate tensors are not aligned, the difference in non-alignment increasing with increasing thermal stability. This result implies further that the 81 components of the tensor  $K_{ijkl}$  defined by the constitutive equations of Eq. (5-13) are not describable by one physical constant of the fluid, such as done in Eq. (5-13a) and what is well satisfied for the coefficients of molecular viscosity (section 5a). On the contrary the eddy-viscosity tensor may depend in a complex way upon the anisotropic character of the turbulent motion.

Equations analogous to Eq. (5-11) may also be applied to turbulent heat and mass transfer. If  $h$  denotes the specific enthalpy then the flux density of eddy heat transport in the equation for the averaged internal energy may be defined (see van Mieghem, 1973) as

$$\vec{I}_h := \overline{gh\vec{u}''} \quad (5-16)$$

Considering in Eq. (5-16) merely sensible heat transfer caused by temperature gradients we may formulate the constitutive relationship for  $\vec{I}_h$  in consistency with Eq. (3-7) as

$$I_h^i = - \sum_j k_h^{ij} \frac{\partial T}{\partial x_j} \quad (i,j = 1,2,3) \quad (5-17)$$

which is relevant for anisotropic turbulent heat exchange and where the components of the tensor  $k_h^{ij}$  denote the thermal eddy-conductivities. If we ignore the anisotropy of this coefficient, i. e. take its isotropic form

$K_h^{ij} = K_h \delta_{ij}$ , then Eq. (5-17) reduces to  $I_h^i = -K_h \partial T / \partial x_i$  ( $i = 1, 2, 3$ ) for all space-directions according to Eq. (6-9).

Considerations of Monin and Yaglom (1971, see sect.10.3) yield that the tensor of Eq. (5-17) can in general be expected to be symmetrical

$$K_h^{ij} = K_h^{ji} \quad ; \quad i,j = 1,2,3 \quad (5-18)$$

which, expressed in another way, equals to the assumption that the components of the heat conductivity tensor are subjected to reciprocal Onsager-conditions. But even in this case we are dealing with six eddy-conductivity coefficients. These coefficients one could not calculate from Eq. (5-17) without any additional hypothesis about the tensorial structure of  $K_h$ . This is the result of the fact that if at a given point in the stream the terms  $\overline{ghu''}$  and  $\partial T / \partial x_i$  are known, it is impossible from the three relations of Eq. (6-17) to define the six different  $K_h^{ij}$ -components.

Note, however, as has been pointed out by Kamenkovich (1967) that the turbulent heat conduction (as well as the turbulent mass diffusion which we do not particularly discuss in the scope of this article), is in a large scale atmospheric and oceanic movement adequately described in terms of a coefficient of horizontal eddy conductivity  $K_h^H$  and a coefficient of vertical eddy conductivity,  $K_h^V$ . Consequently the conductivity tensor may be written in the principal axes formulation inserting  $K_h^H$  for the horizontal elements and  $K_h^V$  for the vertical element within the tensor matrix. We take a particular coordinate system and denote it by  $x, y, z$ : the surface  $x, y$ , is taken as horizontal,  $z$  is directed vertically so that the constitutive equation (5-17) for the vector  $\vec{I}_h$  may be expressed with respect to the  $x, y, z$ -system of coordinates (see also Herbert, 1975)

$$\begin{pmatrix} I_{h,x} \\ I_{h,y} \\ I_{h,z} \end{pmatrix} = - \begin{pmatrix} K_h^H & 0 & 0 \\ 0 & K_h^H & 0 \\ 0 & 0 & K_h^V \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} \quad (5-19)$$

or in symbolic writing

$$\vec{I}_h = -(K_h^V \vec{e}_z \vec{e}_z + K_h^H \delta_H) \cdot \nabla T \quad (5-19a)$$

This relation together with Eq. (4-13) yields that the eddy-conductivity  $K_h = K_h^V \vec{e}_z \vec{e}_z + K_h^H \delta_H$  is a vertically axisymmetric tensor where the coefficients  $A^{VV} + B^{VV}$  and  $B^{VV}$  within Eq. (4-13) are to be replaced by  $K_h^V$  and  $K_h^H$ , respectively. In the study of large-scale atmospheric and oceanic motion it appears to us as natural to assume that the exchange-coefficient of eddy-heat conduction (as well as of eddy-mass diffusion) is axisymmetric relative to a vertical direction. Furthermore this hypothesis of axisymmetry should be applied also within the viscosity coefficients of eddy-frictional processes. Indeed, in large-scale movements it is observed that the momentum along a vertical direction and the momentum along a horizontal surface, differ so considerably that one may ignore any non-isotropy of the eddy-viscosity tensor  $K_{ijkl}$  on a horizontal surface. If we consider the coefficient tensor of eddy viscosity related to the irreversible part of the Reynolds-stress then it becomes, in analogy to Eq. (4-23) under the assumption of axisymmetry, expressible by three scalar coefficients. In this connexion such a formulation should also be considered as an approach with which the discrepancy among the directions of the principal axes of the eddy-stress and the strain rate tensor (discussed in this section above) could be avoided.

## 6 The Constitutive Equations with Axisymmetric Tensor Coefficients

### a Governing Equations

The constitutive equations of heat and friction fluxes represented in Eqs. (3-5) to (3-8) with totally anisotropic coefficients can now be reformulated with axisymmetric coefficients if all partial results calculated in section 4 are applied. Similar to the shear viscosity coefficients  $K_H, K_{HH}$  and  $K_{HV}$  we wish to express all remaining coefficients which couple

fluxes and forces of the same order and character by means of meteorologically familiar notations. Instead of the parameters  $A^{VV}$  and  $B^{VV}$  we will use the coefficients  $K_h^H = B^{VV}$ ,  $K_h^V = A^{VV} + B^{VV}$  where  $K_h^H$  and  $K_h^V$  indicate the heat conductivities in horizontal and vertical directions. Correspondingly we take, instead of the parameters  $K_r^{SS}, A^{aa}$  and  $B^{aa}$ , the coefficients  $K_r^V = K_r^{SS}$ , and  $K_r^H = B^{aa}$  and  $K_r^V = A^{aa} + B^{aa}$  indicating the volume viscosity as well as the rotational friction coefficients in vertical and horizontal directions. Consider that all cross-coefficients coupling fluxes and forces of different order or character shall retain the notation given in section 4.

For the case in which the axis of symmetry is not explicitly specified we then obtain as linear constitutive equations in tensor formulation

$$f = -K_v \nabla \cdot \vec{u} - \mathcal{L}^{SV} \vec{\lambda} \cdot \nabla T - A^{St} (\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta) : (\nabla \vec{u})^{0,S} \quad (6-1)$$

$$\vec{I}_h = -\mathcal{L}^{VS} \vec{\lambda} \nabla \cdot \vec{u} - \left\{ (K_h^V - K_h^H) \vec{\lambda} \vec{\lambda} + K_h^H \delta \right\} \cdot \nabla T + P^{Va} \vec{\lambda} \times (\nabla \times \vec{u} - 2\vec{\omega}) \quad (6-2)$$

$$\vec{I}^a = -\left\{ (K_r^V - K_r^H) \vec{\lambda} \vec{\lambda} + K_r^H \delta \right\} \cdot (\nabla \times \vec{u} - 2\vec{\omega}) + P^{aV} \vec{\lambda} \times \nabla T \quad (6-3)$$

$$I^{0,S} = -A^{ts} (\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta) \nabla \cdot \vec{u} - K_H (\nabla \vec{u})^{0,S} - \left[ \frac{K_H - K_{HH}}{2} \delta \vec{\lambda} \vec{\lambda} + \frac{K_{HV} - K_H}{2} (\vec{\lambda} \vec{e} \vec{\lambda} \vec{e} + \vec{e} \vec{\lambda} \vec{\lambda} \vec{e} + \vec{\lambda} \vec{e} \vec{e} \vec{\lambda} + \vec{e} \vec{\lambda} \vec{e} \vec{\lambda}) + \frac{1}{2} (K_H + 3K_{HH} - 4K_{HV}) \vec{\lambda} \vec{\lambda} \vec{\lambda} \vec{\lambda} \right] : (\nabla \vec{u})^{0,S} \quad (6-4)$$

where the vector  $\vec{e}$  has been supplied to denote the unit vectors in the directions of the cartesian axes of coordinates. If we assume that the axis of symmetry coincides with the vertical direction of a cartesian system of coordinates, we readily find, after inserting  $\vec{\lambda} = \vec{e}_z$  together with the partial results obtained in section 4a to d, that the linear constitutive equations quoted in Eqs (6-2) to (6-4) above may be expressed as follows

$$\begin{aligned} \vec{f} = & -K_v \nabla \cdot \vec{u} - \mathcal{L}^{sv} \frac{\partial T}{\partial z} - \\ & - \frac{A^{ts}}{3} (2\vec{e}_z \vec{e}_z - \delta_H) : (\nabla \vec{u})^{o,s} \end{aligned} \quad (6-5)$$

$$\begin{aligned} \vec{I}_h = & -\mathcal{L}^{vs} \vec{e}_z \nabla \cdot \vec{u} - \\ & -K_h^v \frac{\partial T}{\partial z} \vec{e}_z - K_h^H \delta_H \cdot \nabla_H T + \\ & + \rho^{va} \left[ \nabla \omega - \frac{\partial \vec{u}}{\partial z} - 2(\omega_x \vec{e}_x - \omega_y \vec{e}_y) \right] \end{aligned} \quad (6-6)$$

$$\begin{aligned} \vec{I}^a = & - \\ & - (K_r^v \vec{e}_z \vec{e}_z + K_r^H \delta_H) \cdot (\nabla \times \vec{u} - 2\vec{\omega}) + \\ & + \rho^{av} \left( \frac{\partial T}{\partial x} \vec{e}_x - \frac{\partial T}{\partial y} \vec{e}_y \right) \end{aligned} \quad (6-7)$$

$$\begin{aligned} \vec{I}^{o,s} = & - \frac{A^{ts}}{3} (2\vec{e}_z \vec{e}_z - \delta_H) \nabla \cdot \vec{u} - \\ & - \frac{K_H}{2} (\nabla_H \vec{u}_H + \vec{u}_H \nabla_H - \nabla \cdot \vec{u}_H \delta_H) - \\ & - \frac{K_{HH}}{2} \left( \frac{\nabla \cdot \vec{u}}{3} - \frac{\partial \omega}{\partial z} \right) (\delta_H - 2\vec{e}_z \vec{e}_z) - \\ & - \frac{K_{HV}}{2} \left[ \vec{e}_z \left( \frac{\partial \vec{u}_H}{\partial z} + \nabla_H \omega \right) + \left( \frac{\partial \vec{u}_H}{\partial z} + \nabla_H \omega \right) \vec{e}_z \right] \end{aligned} \quad (6-8)$$

$$\text{with } \nabla_H = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y}, \quad \vec{u}_H = \vec{e}_x u + \vec{e}_y v$$

It should be noted that both versions of the constitutive equations (Eqs. 6-1 to 6-8) demonstrate instructively the typical properties of

anisotropic systems contrasting those of isotropic systems (see Eq. (5-9)) in which each flux depends only on one single scalar coefficient. Despite the extremely strong mathematical resemblance between axisymmetry and isotropy discussed in section 5a it will be obvious from the structure of the constitutive equations that axisymmetric coefficients are able to describe in much more detail the physical nature of turbulent atmospheric (or oceanic) transport phenomena.

Let us consider the so-called simple phenomena of heat, momentum and angular-momentum transfer within the axisymmetric fluid. Here the vectorial fluxes are expressed in terms of two exchange coefficients each (see also table 3), i. e. in that manner we could readily imagine in section 5b, Eqs. (5-19, 19a) where the flux density of enthalpy and sensible heat were discussed. Hitherto we must note that the coefficients of rotational viscosity  $K_r^H$  and  $K_r^V$  have not yet been examined either in atmospheric or oceanic studies since a flux density  $\vec{I}^a$  which may depend on vortex-vector  $\nabla \times \vec{u}$  and intrinsic macroscopic angular frequency  $\vec{\omega}$  was always ignored. Based on the theory treated in the present article it is believed, however, that the rotational exchange of friction and the vortex-effect would constitute an important contribution to the completeness of the equations of motion; this completeness would be very meaningful for the treatment of small-scale irreversible friction processes which must be considered in a series of specific dynamic problems as well as in general circulation models and long range prediction of the atmospheric field parameters.

From Eqs. (6-4) or (6-8) we further recognize that the shear-frictional flux is related to the tensor of strain through the three viscosity coefficients  $K_H$ ,  $K_{HH}$  and  $K_{HV}$ . We thus conclude that the directions of the principal axes of stress and strain rate tensor are different from each other. In our discussion of section 5b this behaviour was argued to be characteristic for eddy momentum transfer in atmosphere and ocean. Recent studies to this

problem utilized by Fiedler (1975) on the basis of velocity observations within the turbulent boundary layer document qualitatively the non-alignment of the principal axes of the tensors of stress and strain.

With regard to the cross-processes due to axisymmetric exchange it is remarkable that the vectorial flux densities  $\vec{I}_h, \vec{I}^a$  (describing heat transfer as well as momentum transfer by rotational friction) and the shear-frictional flux density  $I^{o,s}$  as well as the bulk-viscous flux  $f$  and  $\vec{I}^a$  do not interfere (see also table 3) such as is quite typical in isotropic systems. All remaining cross-processes, however must be supposed to take place explicitly declaring the anisotropic character of the flow. But note also the interesting fact that along the axis of symmetry the cross-processes between the polar flux vector  $\vec{I}_h$  and the axial flux vector  $\vec{I}^a$  disappear (see Eqs.(6-12/13)) which, on the other hand, shows also weak resemblance to the structure of the constitutive equations of isotropic systems.

Concerning the coherence of the cross-effects it must be noted that the 6 correlated cross-coefficients are not independent of each other if their corresponding fluxes of heat, momentum

and angular-momentum are assumed to be reciprocally correlated in the meaning of the Onsager-Casimir principle. Then, according to the reciprocal relationships the coefficients (see e.g. Haase, 1963; Gyarmati, 1970)  $\mathcal{L}^{SV}$  and  $\mathcal{L}^{VS}$  as well as  $\mathcal{P}^{VA}$  and  $\mathcal{P}^{AV}$  are coefficients of Casimir-type while  $A^{St}$  and  $A^{tS}$  are coefficients of Onsager-type. This implies the conditions

$$\mathcal{L}^{VS} = -\mathcal{L}^{SV}, \quad \mathcal{P}^{AV} = -\mathcal{P}^{VA} \quad (6-9)$$

$$A^{tS} = +A^{St} \quad (6-10)$$

The total number of coefficients remaining to be determined in the axisymmetric case reduces from 14 to 11. These are summarized with respect to character and meaning in Table 3. Finally, as a meaningful result of the axisymmetric considerations it should be remarked that all 256 components of the exchange tensors appearing within the original equations (3-4) to (3-8) are representable in terms of those 11 parameters. Their dependence on the properties of the fluid and the flow must be determined by means of experiments or with the aid of statistical methods and theories. From this view the coefficients are constants of state.

|                                  | heat conduction    | volume viscosity | rotational friction | shear friction                | volume viscosity heat conduction       | heat conduction rotational friction    | volume viscosity shear friction |
|----------------------------------|--------------------|------------------|---------------------|-------------------------------|--|--|---------------------------------|
| coefficients of simple processes | $K_h^H$<br>$K_h^V$ | $K_v$            | $K_r^H$<br>$K_r^V$  | $K_H$<br>$K_{HH}$<br>$K_{HV}$ | —                                      | —                                      | —                               |
| coefficients of cross-processes  | —                  | —                | —                   | —                             | $\mathcal{L}^{SV} = -\mathcal{L}^{VS}$ | $\mathcal{P}^{AV} = -\mathcal{P}^{VA}$ | $A^{tS} = +A^{St}$              |

Table 3: Axisymmetric constitutive coefficients

b Constitutive Equations For The z-Direction

It is known that for many special investigations of the atmosphere's dynamic structure the vertical exchange of mass, momentum, angular-momentum and energy is of a particular interest. The constitutive equations parametrizing the vertical components of the corresponding flux densities can easily be found from Eqs. (6-1) to (6-8) by means of the vector operation  $\vec{e}_z \cdot (\dots)$  and considering thereby that the structures of the scalar equations (6-1) or (6-4) evidently remain intact. For convenience sake we only consider here the case in which the axis of symmetry is vertical. From Eqs. (6-5) to (6-8) we then find the following relations for the vertical flux components

$$I_h = -\mathcal{L}^{sv} \nabla \cdot \vec{u} - K_h^v \frac{\partial T}{\partial z} \quad (6-11)$$

$$I^a = -K_r \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 2\omega_z \right) \quad (6-12)$$

$$\vec{\tau} = -\frac{K_{HV}}{2} \frac{\partial \vec{u}_H}{\partial z} - \left\{ \frac{1}{3} (2A^{st} - K_{HH}) \nabla \cdot \vec{u} - K_{HH} \frac{\partial \omega}{\partial z} \right\} \vec{e}_z \quad (6-13)$$

In addition we will consider that vertical fluxes are mainly applied to describe exchange processes under the conditions of the turbulent air layer between earth surface and free atmosphere where the influence of volume viscosity fluxes are usually omitted. Omitting also all cross-effects in Eqs. (6-11) and (6-13) coupled with the existence of volume viscosity we obtain that the vertical transfer of heat, rotational and shear friction may be described by the reduced set of constitutive equations

$$I_h = -K_h^v \frac{\partial T}{\partial z} \quad (6-14)$$

$$I^a = -K_r^v \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 2\omega_z \right) \quad (6-15)$$

$$\vec{\tau} = -\frac{K_{HV}}{2} \frac{\partial \vec{u}_H}{\partial z} - K_{HH} \left( \frac{\partial \omega}{\partial z} - \frac{1}{3} \nabla \cdot \vec{u} \right) \vec{e}_z \quad (6-16)$$

Here  $\vec{\tau}$  is the so-called shear-stress vector defined by  $\vec{\tau} = \vec{e}_z \cdot I^{0,s}$  and consisting of the three components  $(I_{zx}^{0,s}, I_{zy}^{0,s}, I_{zz}^{0,s})$ . Note that  $\vec{\tau}$  still depends on two coefficients of which the coefficient  $K_{HV}$  belongs to the tangential stress vector  $\vec{\tau}_H = (I_{zx}^{0,s}, I_{zy}^{0,s}, 0)$  while the coefficient  $K_{HH}$  belongs to the vertical component of the stress-vector  $I_{zz}^{0,s}$ . Consequently we may Eq. (6-16) equivalently separate into the two portions

$$I_{zz}^{0,s} = -\frac{K_{HH}}{3} \left( 2 \frac{\partial \omega}{\partial z} - \nabla_H \cdot \vec{u}_H \right) \quad (6-17)$$

$$\vec{\tau}_H = -\frac{K_{HV}}{2} \frac{\partial \vec{u}_H}{\partial z}$$

where  $\nabla_H \cdot \vec{u}_H = \partial u / \partial x + \partial v / \partial y =$  horizontal velocity divergence. We wish to emphasize that the constitutive equations (6-14) to (6-17) represent the simplest flux-force relations which without requiring any further physical restrictions can be written down with respect to the z-direction of a system of coordinates, the axis of this direction coinciding with the axis of symmetry. In this connexion it should be of a particular meteorological interest that there exists beside a vertically directed heat flux  $I_h$  also two vertically directed friction fluxes of which (i) the rotational friction flux  $I^a$  belonging to the antisymmetric part of the stress tensor contains the vorticity  $\partial v / \partial x - \partial u / \partial y$  as driving force and  $K_r^v$  as exchange coefficient, (ii) the shear friction flux  $I_{zz}^{0,s}$  belonging to the symmetric part of the stress tensor contains the horizontal and vertical velocity divergence  $\nabla_H \cdot \vec{u}_H$  and  $\partial \omega / \partial z$  as driving forces and  $K_{HH}$  as exchange coefficient. Note that such fluxes have never been taken into account when frictional exchange processes within the turbulent atmospheric surface layer were considered.

Those constitutive relations which are usually applied to boundary layer exchange can be verified if we introduce in the equations of this section the famous Prandtl layer assumptions meaning that the fluid behaves homogeneously with respect to a horizontal surface and that the vertical velocity is negligible. These conditions, i. e.  $\partial / \partial x = \partial / \partial y = 0$  and  $\omega = 0$ , then

yield the vanishing of the vorticity as well as of all components of velocity divergence so that follows  $I^a = 2K_Y^V \omega_z$  and  $I_{zz}^{0,s} = 0$ . Ignoring also  $\omega_z$  and thus  $I^a$  one obtains as remaining constitutive equations

$$I_h = -K_h^V \frac{\partial T}{\partial z}, \quad \vec{\tau}_H = -\frac{K_{HV}}{2} \frac{\partial \vec{u}_H}{\partial z} \quad (6-18)$$

which are in this or somewhat modified formulations usually applied to the eddy-transport of heat and momentum in the atmosphere near the ground. Note that these equations no longer allow any recognition of axisymmetric (or anisotropic) behaviour of the eddying motion. This rather justifies that under the Prandtl layer conditions the irreversible processes of small-scale eddy-heat as well as momentum exchange will be found in dependence on their thermodynamic forces  $\partial T/\partial z$  and  $\partial \vec{u}_H/\partial z$  such as it is usually done in isotropic systems (see section 5a).

## 7 Conclusion

Constitutive equations for linear and angular momentum transfer processes have been derived under conditions applicable to irreversible subgrid-scale exchange in the atmosphere and ocean. A number of aspects have been dealt with in this investigation which so far have been inadequately treated in the meteorological literature. In conclusion, some of the main results will be summarized:

(a) The mechanical bulk- and shear friction transport will have to be completed by means of an antisymmetric part of the stress tensor which causes the exchange between external and intrinsic angular momentum. The antisymmetric stress component can be parameterized in terms of a linear constitutive equation. Therein, the rotation of the velocity vector as well as the intrinsic frequency act as thermodynamic driving forces. Thus the vorticity assumes the meaning of a driving force for the total vertical momentum transport.

(b) It stands to reason that, because of the effect of buoyancy, the mass, energy and momentum transport should be formulated under the condition of axisymmetry. Because of this and

by means of additional natural invariance conditions, the number of independent exchange coefficients can be considerably reduced; the constitutive transport equations retain a structure for an adequate description of atmospheric (and oceanic) exchange.

In particular, from this structure very illuminating derivations were given for the wellknown transport equations of isotropic exchange as well as for the simplified flux-gradient relations usually applied to planetary boundary layer exchange.

(c) The main conclusion, associated with the axisymmetric coefficient structure, are: Cross-effects between shear friction transfer and heat diffusion, shear friction and rotational friction transfer are definitely excluded. Despite this the remaining transport formulation are still general enough to be applicable to extended theories of turbulent motion of the atmosphere such as in the air layer near the ground.

For simple diffusion of heat (corresponding statements are also valid for water vapor) there result two constitutive coefficients (vertical and horizontal) in agreement with the conventional empiric formulations. We obtain completely new contributions to the transport in terms of the horizontal and vertical rotational momentum transfer. Implicitly included are contributions for heat transport (cross-processes) which act perpendicularly to the axis of symmetry. This heat transport as well as the complete inverse rotational friction transport, due to temperature inhomogeneities, are each merely correlated with a single transport coefficient.

Finally, we obtain for the momentum transfer by shear friction processes, in dependency of the deformation field, three coefficients for the description of the fourth ranked viscosity tensor. It follows that the conjugated constitutive equations are applicable for the treatment of arbitrary flow fields in contrast to the conventional shear stress parameterizations which ordinarily apply only to incompressible fluids of zero vertical velocity components.

## 8 Special Considerations to the Axisymmetric Turbulent Exchange of Heat and Momentum - A Note -

### a Introduction

In supplementing the previous developments of sections 1-7 we wish to discuss a special treatment of turbulent momentum and heat exchange. We make the usual assumption that turbulent exchange processes in the atmosphere are caused by fluctuations of the translatory velocity field. Using these special conditions it is possible to assume that the frictional stress tensor is symmetric. Thus it is not required to consider the balance of external and intrinsic angular momentum.

Applying the usual averaging procedure of atmospheric turbulence to the thermo-hydrodynamic relations results in balance equations for the average momentum, mass and energies which are derived by various versions in the meteorological literature. It is our aim in this section to formulate the basic equations in such a manner that the thermodynamic laws can be applied without internal inconsistencies in the presence of small turbulent transport processes. Analogous to the procedures of previous sections, the viscous and the heat flux can be parameterized with the aid of the entropy balance equation and the assumption of axisymmetric exchange.

For practical purposes we will present in section 8c the parameterized axisymmetric divergence expressions of the heat and momentum fluxes in general orthogonal and special coordinates.

<sup>1</sup> The equation of state for air (as perfect gas) then may be written  $\bar{\pi} = \bar{\rho} R_0 \bar{T}$ , with  $R_0 = 287.05 \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$ .

### b Thermodynamics Of A One-Component Turbulent Viscous Fluid

#### Basic Concepts

i We assume that thermodynamic equilibrium states exist also when turbulent processes take place. The conjugate thermodynamic variables then are average-values of pressure, temperature, energy etc. As temperature in the turbulent system we utilize the quantity  $\bar{T}$  which is defined by the thermal equation of state  $F(\bar{\pi}, \bar{\rho}, \bar{T}) = 0$ <sup>1</sup> from which  $\bar{T}$  is computable as function of the thermodynamic pressure  $\bar{\pi}$  and the density of the total system  $\bar{\rho}$ .

ii As analytical form of the first law of thermodynamics we will utilize the balance equation for the mean internal energy. This equation can be derived from the independent balances for the mean total energy and all mechanical energy types. Thus, in absence of intrinsic rotations of the system we can define the total mean energy to be the sum of the potential energy, the kinetic energies of the mean and of the eddying motion as well as the mean internal energy:

$$\hat{E} = \hat{\Phi} + \hat{E}_T + E + \hat{E}_I \quad (8-1)$$

where each energy expression is referred to unit mass.

iii To apply the laws of classical thermodynamics to the variables of the mean state of turbulent systems it is particularly important to give a suitable definition of the mean total internal energy per unit mass  $\hat{E}_{I,tot}$ . Setting  $\hat{E}_{I,tot} = \hat{E}_I$  (Herbert, 1975; Hinkelmann, 1973) is disadvantageous since the thermodynamic basic principles cannot be formulated without some internal inconsistencies. To avoid such problems we define

$$\hat{E}_{I,tot} = \hat{E}_I + E = \hat{E} - \hat{\Phi} - \hat{E}_T \quad (8-2)$$

According to Eq. (8-2) and to molecular-kinetic relations we require that the total mean ther-

dynamic pressure consists of

$$\bar{\pi} = \bar{p} + \frac{1}{3} \overline{\rho \vec{u}''^2} = \bar{p} + \frac{2}{3} \bar{\rho} E \quad (8-3)$$

where  $\bar{p}$  is the mean molecular pressure and  $2\bar{\rho} E$  is the trace of the complete Reynolds stress tensor,  $2\bar{\rho} E = \text{Tr} \overline{\rho \vec{u}'' \vec{u}''}$

iii Utilizing Eqs. (8-2, 3) the Gibbs basic equation may then be formulated as

$$\int \hat{\epsilon}_{I, \text{tot}} = \tilde{T} \delta \hat{S} - \bar{\pi} \delta (1/\bar{\rho}) \quad (8-4)$$

where  $\hat{S}$  is the mean entropy and  $1/\bar{\rho} = \hat{v}$ . Eq. (8-4) may be thought of as a theorem and as an analytical formulation of the second law of thermodynamics for a one-component turbulent system.

### Energetics

Using Eq. (8-3) the equation of mean motion for the system of the rotating earth may be written as

$$\begin{aligned} \bar{\rho} \frac{D \hat{\vec{u}}}{Dt} + \nabla \cdot (\bar{\mathbf{I}} + \mathbf{I}_{Re}) = \\ = -\nabla \bar{\pi} - \bar{\rho} \nabla \Phi - 2\bar{\rho} \vec{\Omega} \times \hat{\vec{u}} \end{aligned}$$

where  $\hat{\vec{u}}$  = mean velocity vector,  $\bar{\mathbf{I}}$  = mean molecular friction tensor,  $\mathbf{I}_{Re} = \overline{\rho \vec{u}'' \vec{u}''} - \frac{2}{3} \bar{\rho} E$  (c.f. Eq. 5-11) = symmetric and traceless part of the Reynolds-stress tensor and  $\vec{\Omega}$  = the constant earth's angular frequency. Scalar multiplication of the equation of motion by  $\hat{\vec{u}}$  results in the budget equation for the mean kinetic energy  $\hat{\epsilon}_T = \hat{\vec{u}}^2/2$  :

$$\begin{aligned} \bar{\rho} \frac{D \hat{\vec{u}}^2/2}{Dt} + \nabla \cdot (\bar{\pi} \delta + \bar{\mathbf{I}} + \mathbf{I}_{Re}) \cdot \hat{\vec{u}} = \\ = \bar{\pi} \nabla \cdot \hat{\vec{u}} + \bar{\mathbf{I}} : \nabla \hat{\vec{u}} + \\ + \mathbf{I}_{Re} : \left[ \frac{1}{2} (\nabla \hat{\vec{u}} + \hat{\vec{u}} \nabla) - \frac{1}{3} \nabla \cdot \hat{\vec{u}} \delta \right] \end{aligned} \quad (8-5)$$

In addition, the budget equations for the potential energy  $\Phi$  and the total energy  $\hat{\epsilon}$  are

given by

$$\left. \begin{aligned} \bar{\rho} \frac{D \Phi}{Dt} = \bar{\rho} \hat{\vec{u}} \cdot \nabla \Phi \\ \bar{\rho} \frac{D \hat{\epsilon}}{Dt} + \nabla \cdot \hat{\mathbf{I}}_{\hat{\epsilon}} = 0 \end{aligned} \right\} \quad (8-6)$$

According to Eqs (8-2, 5 and 6) the first law of thermodynamics is then written in terms of the budget equation of the total mean internal energy as follows

$$\begin{aligned} \bar{\rho} \left( \frac{D \hat{\epsilon}_{I, \text{tot}}}{Dt} + \bar{\pi} \frac{D 1/\bar{\rho}}{Dt} \right) = \\ = -\nabla \cdot \hat{\mathbf{I}}_h - \bar{\mathbf{I}} : \nabla \hat{\vec{u}} - \\ - \mathbf{I}_{Re} : \left[ \frac{1}{2} (\nabla \hat{\vec{u}} + \hat{\vec{u}} \nabla) - \frac{1}{3} \nabla \cdot \hat{\vec{u}} \delta \right] \end{aligned} \quad (8-7)$$

Therein we abbreviated  $\hat{\mathbf{I}}_{\hat{\epsilon}} - (\bar{\pi} \delta + \bar{\mathbf{I}} + \mathbf{I}_{Re}) \cdot \hat{\vec{u}}$  by  $\hat{\mathbf{I}}_h$  describing the total heat flux density of the system.

In the following analysis the molecular viscosity flux shall be ignored in order to focus attention on the fluxes  $\hat{\mathbf{I}}_h$  and  $\mathbf{I}_{Re}$ . Then follows from Eqs. (8-4 and 7) and the continuity equation the budget equation of entropy as

$$\begin{aligned} \bar{\rho} \frac{D \hat{S}}{Dt} + \nabla \cdot \left( \frac{\hat{\mathbf{I}}_h}{\tilde{T}} \right) = - \frac{\hat{\mathbf{I}}_h \cdot \nabla \tilde{T}}{\tilde{T}^2} - \\ - \frac{1}{2\tilde{T}} \mathbf{I}_{Re} : \left( \nabla \hat{\vec{u}} + \hat{\vec{u}} \nabla - \frac{2}{3} \nabla \cdot \hat{\vec{u}} \delta \right) \geq 0 \end{aligned} \quad (8-8)$$

The right side of Eq. (8-8) represents the entropy source strength function which is zero at equilibrium,  $\hat{\mathbf{I}}_h = 0$ ,  $\mathbf{I}_{Re} = 0$ , and which is greater than zero at non-equilibrium.

At this point we especially note that as a consequence of former definitions only the deviator of the total Reynolds stress tensor occurs in the formulation of the irreversible turbulent viscosity flux; the reason is that the trace of the complete Reynolds stress tensor acts as part of the thermodynamic pressure. On the other hand, if one defines  $\bar{\pi} = \bar{p}$ , it becomes necessary to modify the former theory so that in Eq. (8-8) the deviator  $\mathbf{I}_{Re}$  must be replaced by the total stress tensor  $\overline{\rho \vec{u}'' \vec{u}''}$ .

Such a definition should be avoided in as much as this implies the physical defect that now represents some type of bulk-viscosity.

Constitutive Equations

As shown in section 5b and 6 a realistic approach for the treatment of atmospheric exchange processes is to take the exchange coefficients as symmetric with respect to an invariant axis. One usually assumes that the axis of symmetry coincides with the vertical direction of a cartesian coordinate system. With this prerequisite,  $\hat{I}_h$  and  $I_{Re}$  can be parameterized according to Eqs. (6-6 to 6-8); note well however, that in these equations the flux components coupled to the coefficients  $L^{vs}$ ,  $P^{va}$  and  $A^{ts}$  do not occur under the special conditions of section 8. The flux components then take the form

$$\begin{aligned} \hat{I}_{h,x} &= -k_{h,H} \frac{\partial \tilde{T}}{\partial x} \\ \hat{I}_{h,y} &= -k_{h,H} \frac{\partial \tilde{T}}{\partial y} \\ \hat{I}_{h,z} &= -k_{h,V} \frac{\partial \tilde{T}}{\partial z} \end{aligned}$$

(8-9)

$$\begin{aligned} I_{Re,xx} &= -k_H U_{xx} - \frac{k_H - k_{HH}}{2} U_{zz} \\ &\quad - \frac{k_H + k_{HH}}{2} U_{xx} - \frac{k_{HH}}{2} U_{yy} \end{aligned}$$

$$\begin{aligned} I_{Re,yy} &= -k_H U_{yy} - \frac{k_H - k_{HH}}{2} U_{zz} \\ &\quad - \frac{k_H + k_{HH}}{2} U_{yy} - \frac{k_{HH}}{2} U_{xx} \end{aligned}$$

$$\begin{aligned} I_{Re,zz} &= -k_{HH} U_{zz}, I_{Re,xy} = I_{Re,yx} = \\ &= -k_H U_{xy}, I_{Re,xz} = I_{Re,zx} = -k_{HV} U_{xz} \\ I_{Re,yz} &= I_{Re,zy} = -k_{HV} U_{yz} \end{aligned}$$

where

$$U_{ij} = \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} - \frac{2}{3} \nabla \cdot \hat{u} \delta_{ij} \quad (i,j=x,y,z)$$

implying the identity  $U_{zz} = -(U_{xx} + U_{yy})$ .

With the use of Eq. (8-9) the corresponding dissipative energy rates then may be computed as

$$W_h = - \sum I_{h,i} \frac{\partial \tilde{T}}{\partial x_i} = k_{h,H} [(\frac{\partial \tilde{T}}{\partial x})^2 + (\frac{\partial \tilde{T}}{\partial y})^2] + k_{h,V} (\frac{\partial \tilde{T}}{\partial z})^2 \geq 0 \quad (8-10)$$

$$\begin{aligned} W_{Re} &= - \sum I_{Re,ij} \frac{U_{ij}}{2} = k_H U_{xy}^2 + \\ &+ k_{HV} (U_{xz}^2 + U_{yz}^2) + \frac{3k_{HH} k_H}{3k_{HH} + k_H} U_{yy}^2 + \\ &+ \frac{1}{4} (3k_{HH} + k_H) (U_{xx} + \frac{3k_{HH} - k_H}{3k_{HH} + k_H} U_{yy})^2 \geq 0 \end{aligned} \quad (8-11)$$

For the two forms  $W_h$  and  $W_{Re}$  to be positive at arbitrary values of  $\partial \tilde{T} / \partial x_i$  and  $U_{ij}$ , the following necessary and sufficient conditions

$$\left. \begin{aligned} k_{h,H} > 0, k_{h,V} > 0, \\ k_H > 0, k_{HH} > 0, k_{HV} > 0 \end{aligned} \right\} \quad (8-12)$$

must be obeyed.

c Divergences Of Heat And Momentum Fluxes

Basic relations

For the purpose of practical applications we will formulate the flux divergences of  $\hat{I}_h$  and  $I_{Re}$  in general and special orthogonal coordinate systems. Utilizing a contragredient mathematical analysis we obtain from the transformation rules that an orthogonal system of arbitrary contravariant coordinates  $q_1, q_2, q_3$  is completely described by the scale factors

$$H_i = \sqrt{\left( \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_i} \right)}_{i, \text{no sum} (i=1,2,3)}, \text{ with } x_k \text{ denoting cartesian coordinates, as well as by the Jakobian determinant } \sqrt{g} = H_1 H_2 H_3.$$

In terms of these quantities the divergences of  $\hat{I}_h$  and  $I_{Re}$  may be written as follows

$$\nabla \cdot \hat{I}_h = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_n} \left( \sqrt{g} \frac{\hat{I}_{h,n}}{H_n} \right) \quad (8-13)$$

$$\nabla \cdot I_{Re} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_n} \left( \sqrt{g} \frac{I_{Re,nj} \vec{e}_j}{H_n} \right) \quad (j=1,2,3) \quad (8-14)$$

where  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  denote the unit vectors in direction of the arbitrary coordinates and where  $\hat{I}_{h,k}$  and  $I_{Re,kj}$  thus represent the physical components (not the co- or contravariant ones) of  $\hat{I}_h$  and  $I_{Re}$ , respectively. With the tensorial expression

$$\frac{\partial \vec{e}_i}{\partial q_j} = \frac{\vec{e}_j}{H_i} \frac{\partial H_i}{\partial q_i} - \frac{\vec{e}_k}{H_k} \frac{\partial H_i}{\partial q_k} \delta_{ij}$$

the space derivations of the unit vectors can be eliminated from Eq. (8-10). Thus we obtain

$$\begin{aligned} \nabla \cdot I_{Re} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left( \sqrt{g} \frac{I_{Re,ij}}{H_i} \right) \vec{e}_j + \\ &+ \frac{I_{Re,ij}}{H_i H_j} \frac{\partial H_i}{\partial q_j} \vec{e}_i - \frac{I_{Re,ii}}{H_i H_k} \frac{\partial H_k}{\partial q_k} \vec{e}_k \end{aligned} \quad (8-15)$$

which consists of the three components

$$\begin{aligned} F_k &= \vec{e}_k \cdot (\nabla \cdot I_{Re}) = \\ &= \left[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left( \sqrt{g} \frac{I_{Re,ik}}{H_i} \right) + \frac{I_{Re,ki}}{H_k H_i} \frac{\partial H_k}{\partial q_i} \right. \\ &\quad \left. - \frac{I_{Re,ii}}{H_i H_k} \frac{\partial H_i}{\partial q_k} \right]_{k, \text{no sum}} \quad (k=1,2,3) \end{aligned} \quad (8-16)$$

It should be pointed out that the matrix  $I_{Re,ik}$  may still be non-symmetric in Eq. (8-16). From physical reasoning, however,  $I_{Re,ik}$  must here be assumed to be symmetric.

For the purpose of parameterization the temperature and velocity gradients in the divergence formulas (Eqs. 8-13 and 16) must also be expressed in terms of arbitrary coordinates. In order to represent the components of the heat fluxes  $\hat{I}_{h,i}$  as functions of the temperature gradients in generalized orthogonal coordinates

we must use

$$\begin{aligned} \nabla \tilde{T} &= \frac{\vec{e}_n}{H_n} \frac{\partial \tilde{T}}{\partial q_n} = \\ &= \left( \frac{1}{H_1} \frac{\partial \tilde{T}}{\partial q_1}, \frac{1}{H_2} \frac{\partial \tilde{T}}{\partial q_2}, \frac{1}{H_3} \frac{\partial \tilde{T}}{\partial q_3} \right) \end{aligned} \quad (8-17)$$

In order to express  $U_{ij}$  in terms of general coordinates we take the tensorial form

$$U = \nabla \hat{u} + \hat{u} \nabla - \frac{2}{3} \nabla \cdot \hat{u} \delta$$

which is independent of any special choice of coordinates. According to tensor calculus the gradient and the divergence of  $\hat{u}$  can be written in terms of arbitrary orthogonal coordinates as

$$\begin{aligned} \nabla \hat{u} &= \vec{e}_i \frac{1}{H_i} \frac{\partial \hat{u}}{\partial q_i} \\ \frac{\partial \hat{u}}{\partial q_i} &= \frac{\partial \hat{u}_k}{\partial q_i} \vec{e}_k + \left( \frac{\hat{u}_k}{H_k} \frac{\partial H_i}{\partial q_k} \vec{e}_i \right)_{k \neq i} - \\ &- \left( \frac{\hat{u}_i}{H_s} \frac{\partial H_i}{\partial q_s} \vec{e}_s + \frac{\hat{u}_i}{H_r} \frac{\partial H_i}{\partial q_r} \vec{e}_r \right)_{\text{no sum}, i \neq s \neq r} \\ &= 1,2,3 \\ \nabla \cdot \hat{u} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left( \sqrt{g} \frac{\hat{u}_i}{H_i} \right) \end{aligned}$$

With these expressions  $U_{ik}$  obtains the form:

$$\begin{aligned} U_{ik} &= \frac{H_i}{H_k} \frac{\partial}{\partial q_k} \left( \frac{\hat{u}_i}{H_i} \right) + \frac{H_k}{H_i} \frac{\partial}{\partial q_i} \left( \frac{\hat{u}_k}{H_k} \right) + \\ &+ 2 \delta_{ik} \left[ \frac{\hat{u}_n}{H_i H_n} \frac{\partial H_i}{\partial q_n} - \frac{1}{3\sqrt{g}} \frac{\partial}{\partial q_n} \left( \frac{\sqrt{g} \hat{u}_n}{H_n} \right) \right] \quad (8-18) \\ &(i,k=1,2,3) \end{aligned}$$

### Special cases

Cartesian and stereographic cylindrical coordinates

i In cartesian  $x, y, z$ -coordinates  $\sqrt{g} = 1$  and  $H_1 = H_2 = H_3 = 1$  so that Eq. (8-9) is valid. Assuming for atmospheric conditions that  $k_{n,H} = \text{const}$  and  $k_H = \text{const}$  the heat flux divergence and the frictional flux divergence components  $F_x, F_y$  and  $F_z$  take the form

$$\nabla \cdot \hat{\mathbf{I}}_h = -\frac{\partial}{\partial z} \left( K_{h,v} \frac{\partial \hat{T}}{\partial z} \right) - K_{h,H} \left( \frac{\partial^2 \hat{T}}{\partial x^2} + \frac{\partial^2 \hat{T}}{\partial y^2} \right) \quad (8-19)$$

$$F_x = -K_H \left( \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} \right) - \frac{\partial}{\partial z} \left[ K_{HV} \left( \frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{w}}{\partial x} \right) \right] + \frac{\partial}{\partial x} \left[ K_{HH} \left( \frac{\partial \hat{w}}{\partial z} - \frac{\nabla \cdot \hat{\mathbf{u}}}{3} \right) \right] \quad (8-20)$$

$$F_z = -\frac{\partial}{\partial x} \left[ K_{HV} \left( \frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{w}}{\partial x} \right) \right] - \frac{\partial}{\partial y} \left[ K_{HV} \left( \frac{\partial \hat{v}}{\partial z} + \frac{\partial \hat{w}}{\partial y} \right) \right] - \frac{\partial}{\partial z} \left[ 2K_{HH} \left( \frac{\partial \hat{w}}{\partial z} - \frac{\nabla \cdot \hat{\mathbf{u}}}{3} \right) \right]$$

For incompressible fluids ( $\nabla \cdot \hat{\mathbf{u}} = 0$ ) we obtain instead of Eq. (8-20)

$$F_x = -K_H \left( \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} \right) - \frac{\partial}{\partial z} \left[ K_{HV} \left( \frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{w}}{\partial x} \right) \right] + \frac{\partial}{\partial x} K_{HH} \frac{\partial \hat{w}}{\partial z} \quad (8-21)$$

$$F_z = -\frac{\partial}{\partial x} \left[ K_{HV} \left( \frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{w}}{\partial x} \right) \right] - \frac{\partial}{\partial y} \left[ K_{HV} \left( \frac{\partial \hat{v}}{\partial z} + \frac{\partial \hat{w}}{\partial y} \right) \right] - 2 \frac{\partial}{\partial z} K_{HH} \frac{\partial \hat{w}}{\partial z}$$

$F_x$  and  $F_y$  from Eq. (8-21) are equivalent to those relations found by Kamenkovich (1967). However, they differ strongly from

$$F_x = -K_H \left( \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} \right) - \frac{\partial}{\partial z} K_{HV} \frac{\partial \hat{u}}{\partial z} \quad (8-22)$$

$$F_y = -K_H \left( \frac{\partial^2 \hat{v}}{\partial x^2} + \frac{\partial^2 \hat{v}}{\partial y^2} \right) - \frac{\partial}{\partial z} K_{HV} \frac{\partial \hat{v}}{\partial z} \quad (8-22)$$

mainly utilized in the literature in analogy to the structure of the heat flux divergence, Eq. (8-19). Note that, for Eqs. (8-22 and 19) to be equivalent, we must require  $K_{HV} = K_{HH} = \text{const}$  or  $K_{HH} = 0$  together with  $\frac{\partial \hat{w}}{\partial x}, \frac{\partial \hat{w}}{\partial y} \ll \frac{\partial \hat{u}}{\partial z}, \frac{\partial \hat{v}}{\partial z}$ , which, however, are rather unrealistic approximations.

Furthermore  $F_z$  is completely ignored in Eq. (8-22); this is acceptable if hydrostatic equilibrium can be assumed. With respect to the terms neglected in  $F_x$  and  $F_y$  of Eq. (8-22) we wish to remark: Although it is difficult to give a correct estimate of the coefficient  $K_{HH}$  in turbulent atmospheric motions we can assume that  $K_H$  and  $K_{HH}$  are of the same order of magnitude. Observations of large scale atmospheric motions and scale analysis suggest that  $K_{HV}$ , which considerably varies with height, is much smaller than  $K_H$ . Their values change in the range of some orders of magnitude and may be approximated by (Kamenkovich, 1967):

$$K_{HV} \approx 1-10^3 \text{ g cm}^{-1} \text{ s}^{-1}$$

$$K_H \approx 10^3-10^6 \text{ g cm}^{-1} \text{ s}^{-1}$$

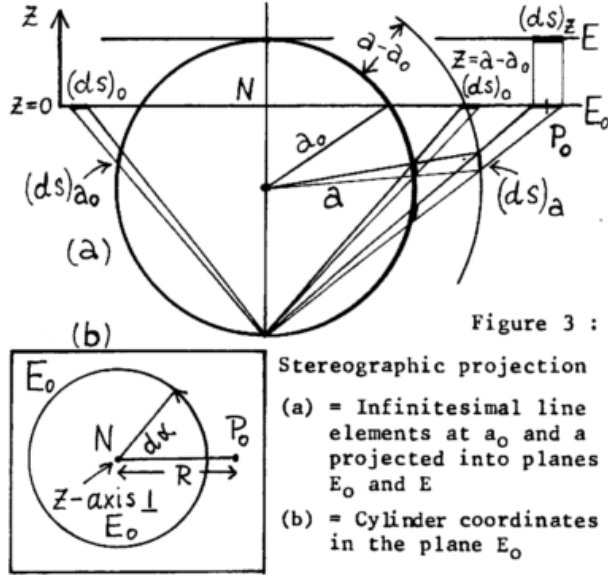
Thus

$$O(K_{HV}) \ll O(K_{HH}) \approx O(K_H) \quad (8-23)$$

indicating that the terms  $\frac{\partial}{\partial x} (K_{HH} \frac{\partial \hat{w}}{\partial z})$  and  $\frac{\partial}{\partial y} (K_{HH} \frac{\partial \hat{w}}{\partial z})$  can be of considerable importance for  $F_x$  and  $F_y$  (Eq. 8-20).

So far we utilized rectilinear cartesian coordinates. A more realistic analysis of large scale atmospheric motions can be carried out in a system of curvilinear coordinates which are adapted to the figure of the earth. Usually one applies spherical coordinates represented in stereographic map projection. For such a projection it is suitable to transform the spherical planes  $a = \text{const}$  into vertically superposed planes parallel to the equatorial plane, where cartesian or cylindric coordinates can be used. A stereographic cylindric system is described by the horizontal coordinates  $R = \sqrt{x^2 + y^2}$ ,  $\alpha$  and the vertical coordinate  $z$ . The planes  $R = \text{const}$  are the cylindric surfaces, the angles

$\alpha = \text{const}$  determine circle segments perpendicular to the  $z$ -axis and  $z = \text{const}$  are circular planes of the distance  $z$ . Note that  $\alpha$  equals the longitude of spherical coordinates and  $z = a - a_0$ , where  $a_0 = \text{mean radius of the earth}$ .



Since stereographic sketches are conformal the square of the line element is given by the quadratic form:

$$ds^2 = \frac{1}{m^2} (dR^2 + R^2 d\alpha^2) + dz^2$$

so that the scale factors become

$$\left. \begin{aligned} H_R &= \frac{1}{m}, H_\alpha = \frac{R}{m}, H_z = 1 \\ \sqrt{g} &= H_1 H_2 H_3 = \frac{R}{m^2} \end{aligned} \right\} \quad (8-24)$$

where  $m$  is the map projection factor depending on  $R$  (or on  $x, y$ ) and on  $z$  but not on  $\alpha$ . For practical purposes the  $z$ -dependency of  $m$  is negligible since the synoptic active height  $z$  is much smaller than the earth's radius  $a_0$ , i. e.  $m = m(R)$ . With Eq. (8-24) then follows for Eqs. (8-13, 16, 17 and 18):

$$\nabla \cdot \hat{I}_h = \frac{m^2}{R} \left\{ \frac{\partial}{\partial R} \left( \frac{R}{m} \hat{I}_{h,R} \right) + \frac{\partial}{\partial \alpha} \left( \frac{1}{m} \hat{I}_{h,\alpha} \right) + \frac{\partial}{\partial z} \left( \frac{R}{m^2} \hat{I}_{h,z} \right) \right\} \quad (8-25)$$

and

$$\frac{\vec{e}_n}{H_n} \frac{\partial \tilde{T}}{\partial q_n} = \left( m \frac{\partial \tilde{T}}{\partial R}, \frac{m}{R} \frac{\partial \tilde{T}}{\partial \alpha}, \frac{\partial \tilde{T}}{\partial z} \right) \quad (8-25a)$$

as well as

$$\begin{aligned} \bar{F}_R &= \frac{m^2}{R} (I_{Re,RR} - I_{Re,\alpha\alpha}) \frac{\partial}{\partial R} \left( \frac{R}{m} \right) + \quad (8-26) \\ &+ m \frac{\partial I_{Re,RR}}{\partial R} + \frac{m}{R} \frac{\partial I_{Re,\alpha R}}{\partial \alpha} + \frac{\partial}{\partial z} I_{Re,zR} \end{aligned}$$

$$\begin{aligned} \bar{F}_\alpha &= \frac{m^2}{R} (I_{Re,R\alpha} + I_{Re,\alpha R}) \frac{\partial}{\partial R} \left( \frac{R}{m} \right) + \\ &+ m \frac{\partial I_{Re,R\alpha}}{\partial R} + \frac{m}{R} \frac{\partial I_{Re,\alpha\alpha}}{\partial \alpha} + \frac{\partial}{\partial z} I_{Re,z\alpha} \end{aligned}$$

$$\bar{F}_z = \frac{m^2}{R} \frac{\partial}{\partial R} \left( \frac{R}{m} I_{Re,Rz} \right) + \frac{m}{R} \frac{\partial I_{Re,\alpha z}}{\partial \alpha} + \frac{\partial I_{Re,zz}}{\partial z}$$

and

$$\begin{aligned} U_{RR} &= \frac{4}{3} \left\{ m \left[ \frac{\partial \hat{U}_R}{\partial R} - \right. \right. \quad (8-26a) \\ &\left. \left. - \frac{1}{2} \left[ \frac{m}{R} \frac{\partial \hat{U}_\alpha}{\partial \alpha} + \frac{\partial \hat{U}_z}{\partial z} + \frac{m^2}{R} \hat{U}_R \frac{\partial R/m}{\partial R} \right] \right\} \end{aligned}$$

$$\begin{aligned} U_{\alpha\alpha} &= \frac{4}{3} \left\{ \frac{m}{R} \left[ \frac{\partial \hat{U}_\alpha}{\partial \alpha} + \right. \right. \\ &\left. \left. + m \hat{U}_R \frac{\partial}{\partial R} \left( \frac{R}{m} \right) - \frac{R}{2} \frac{\partial \hat{U}_R}{\partial R} \right] - \frac{1}{2} \frac{\partial \hat{U}_z}{\partial z} \right\} \end{aligned}$$

$$\begin{aligned} U_{zz} &= \frac{4}{3} \left\{ \frac{\partial \hat{U}_z}{\partial z} - \frac{m}{2R} \left[ m \frac{\partial}{\partial R} \left( \frac{R \hat{U}_R}{m} \right) + \right. \right. \\ &\left. \left. + \frac{\partial \hat{U}_\alpha}{\partial \alpha} \right] \right\} \end{aligned}$$

$$U_{R\alpha} = U_{\alpha R} = \frac{m}{R} \frac{\partial \hat{U}_R}{\partial \alpha} + R \frac{\partial}{\partial R} \left( \frac{m \hat{U}_\alpha}{R} \right)$$

$$U_{Rz} = U_{zR} = \frac{\partial \hat{U}_R}{\partial z} + m \frac{\partial \hat{U}_z}{\partial R}$$

$$U_{\alpha z} = U_{z\alpha} = \frac{\partial \hat{U}_\alpha}{\partial z} + \frac{m}{R} \frac{\partial \hat{U}_z}{\partial \alpha}$$

Let now the invariant axis of symmetry coincide in every point with the  $a$ -axis of spherical coordinates and hence with  $z = a - a_0$  of the stereographic cylinder system. Then from the method of obtaining the constitutive equations

(8-9) and using Eqs. (8-25a, 26a) follow the corresponding constitutive equations in stereographic cylinder coordinates

$$\begin{aligned}\hat{I}_{h,R} &= -K_{h,H} H_R^{-1} \frac{\partial \tilde{T}}{\partial R} = -m K_{h,H} \frac{\partial \tilde{T}}{\partial R} \\ \hat{I}_{h,\alpha} &= -K_{h,H} H_\alpha^{-1} \frac{\partial \tilde{T}}{\partial \alpha} = -\frac{m K_{h,H}}{R} \frac{\partial \tilde{T}}{\partial \alpha} \quad (8-27) \\ \hat{I}_{h,z} &= -K_{h,V} H_z^{-1} \frac{\partial \tilde{T}}{\partial z} = -K_{h,V} \frac{\partial \tilde{T}}{\partial z}\end{aligned}$$

$$\begin{aligned}I_{Re,RR} &= -\frac{K_H + K_{HH}}{2} U_{RR} - \frac{K_{HH}}{2} U_{\alpha\alpha} = \\ &= -\frac{2K_H}{3} \left\{ \frac{m}{\partial R} \frac{\partial \hat{U}_R}{\partial R} - \frac{1}{2} \left[ \frac{m}{R} \frac{\partial \hat{U}_\alpha}{\partial \alpha} + \frac{\partial \hat{U}_z}{\partial z} + \frac{m^2 \hat{U}_R}{R} \frac{\partial R/m}{\partial R} \right] \right\} \\ &\quad - \frac{2K_{HH}}{3} \left\{ \frac{m}{2} \frac{\partial \hat{U}_R}{\partial R} + \frac{m}{2R} \left[ \frac{\partial \hat{U}_\alpha}{\partial \alpha} + m \hat{U}_R \frac{\partial R/m}{\partial R} \right] - \frac{\partial \hat{U}_z}{\partial z} \right\}\end{aligned}$$

$$\begin{aligned}I_{Re,\alpha\alpha} &= -\frac{K_H + K_{HH}}{2} U_{\alpha\alpha} - \frac{K_{HH}}{2} U_{RR} = \\ &= -\frac{2K_H}{3} \left\{ \frac{m}{R} \left[ \frac{\partial \hat{U}_\alpha}{\partial \alpha} + m \hat{U}_R \frac{\partial R/m}{\partial R} - \frac{R \partial \hat{U}_R}{2 \partial R} \right] - \frac{\partial \hat{U}_z}{2 \partial z} \right\} \\ &\quad - \frac{2K_{HH}}{3} \left\{ \frac{m}{2R} \left[ \frac{\partial \hat{U}_\alpha}{\partial \alpha} + R \frac{\partial \hat{U}_R}{\partial R} + m \hat{U}_R \frac{\partial R/m}{\partial R} \right] - \frac{\partial \hat{U}_z}{\partial z} \right\}\end{aligned}$$

$$\begin{aligned}I_{Re,zz} &= -K_{HH} U_{zz} = \quad (8-28) \\ &= -\frac{4K_{HH}}{3} \left\{ \frac{\partial \hat{U}_z}{\partial z} - \frac{m}{2R} \left[ m \frac{\partial}{\partial R} \left( \frac{R \hat{U}_R}{m} \right) + \frac{\partial \hat{U}_\alpha}{\partial \alpha} \right] \right\}\end{aligned}$$

$$\begin{aligned}I_{Re,R\alpha} &= I_{Re,\alpha R} = -K_H U_{R\alpha} = \\ &= -K_H \left[ \frac{m}{R} \frac{\partial \hat{U}_R}{\partial \alpha} + R \frac{\partial}{\partial R} \left( \frac{m \hat{U}_\alpha}{R} \right) \right]\end{aligned}$$

$$\begin{aligned}I_{Re,Rz} &= I_{Re,zR} = -K_{HV} U_{Rz} = \\ &= -K_{HV} \left[ \frac{\partial \hat{U}_R}{\partial z} + m \frac{\partial \hat{U}_z}{\partial R} \right]\end{aligned}$$

$$\begin{aligned}I_{Re,\alpha z} &= I_{Re,z\alpha} = -K_{HV} U_{\alpha z} = \\ &= -K_{HV} \left[ \frac{\partial \hat{U}_\alpha}{\partial z} + \frac{m}{R} \frac{\partial \hat{U}_z}{\partial \alpha} \right]\end{aligned}$$

Inserting Eqs. (8-27) and (8-28) into Eqs. (8-25) and (8-26), respectively, and assuming the horizontal coefficients  $K_{h,H}$  and  $K_H$  to be constant we finally have

$$\begin{aligned}\nabla \cdot \hat{I}_h &= -m^2 K_{h,H} \frac{\partial^2 \tilde{T}}{\partial R^2} - \frac{m^2}{R} K_{h,H} \frac{\partial \tilde{T}}{\partial R} - \\ &\quad - \frac{m^2}{R^2} K_{h,H} \frac{\partial^2 \tilde{T}}{\partial \alpha^2} - \frac{\partial}{\partial z} \left( K_{h,V} \frac{\partial \tilde{T}}{\partial z} \right) \quad (8-29)\end{aligned}$$

$$\begin{aligned}F_R &= -K_H \left\{ \Delta_1 \hat{U}_R - \frac{m^3}{R^2} \frac{\partial R/m}{\partial R} \frac{\partial \hat{U}_\alpha}{\partial \alpha} - \frac{m}{3} \frac{\partial^2 \hat{U}_z}{\partial R \partial z} - \right. \\ &\quad \left. - \frac{m^4}{R^2} \hat{U}_R \left( \frac{\partial R/m}{\partial R} \right)^2 - \frac{m \hat{U}_R}{3} \frac{\partial}{\partial R} \left( \frac{m^2 \partial R/m}{R} \right) + \right. \\ &\quad \left. \frac{2m}{3} \frac{\partial^2 \left( \frac{m \hat{U}_\alpha}{R} \right)}{\partial \alpha \partial R} \right\} + \frac{2}{3} \frac{\partial}{\partial R} \left\{ K_{HH} \left[ \frac{\partial \hat{U}_z}{\partial z} - \frac{m}{2} \left( \frac{\partial \hat{U}_R}{\partial R} + \right. \right. \right.\end{aligned}$$

$$\left. \frac{\partial \hat{U}_\alpha}{R \partial \alpha} + \frac{\hat{U}_R}{R} \frac{\partial R/m}{\partial R} \right\} - \frac{\partial}{\partial z} K_{HV} \frac{\partial \hat{U}_R}{\partial z} - m \frac{\partial}{\partial z} K_{HV} \frac{\partial \hat{U}_z}{\partial R}$$

$$\begin{aligned}F_\alpha &= -K_H \left[ \Delta_2 \left( \frac{\hat{U}_\alpha m}{R} \right) + \Delta_3 \left( \frac{\hat{U}_R m}{R} \right) + \frac{m}{3R} \frac{\partial^2 \hat{U}_z}{\partial \alpha \partial z} \right] \\ &\quad - \frac{2m}{3R} \left\{ \frac{m}{2R} \frac{\partial}{\partial \alpha} \left( K_{HH} \frac{\partial \hat{U}_\alpha}{\partial \alpha} \right) + \frac{m}{2} \frac{\partial}{\partial \alpha} \left( K_{HH} \frac{\partial \hat{U}_R}{\partial R} \right) \right. \\ &\quad \left. + \frac{m^2}{2R} \frac{\partial R/m}{\partial R} \frac{\partial}{\partial \alpha} \left( K_{HH} \hat{U}_R \right) - \frac{\partial}{\partial \alpha} \left( K_{HH} \frac{\partial \hat{U}_z}{\partial z} \right) \right\} - \\ &\quad - \frac{\partial}{\partial z} \left( K_{HV} \frac{\partial \hat{U}_\alpha}{\partial z} \right) - \frac{m}{R} \frac{\partial}{\partial z} \left( K_{HV} \frac{\partial \hat{U}_z}{\partial \alpha} \right) \quad (8-30)\end{aligned}$$

$$\begin{aligned}F_z &= -\frac{m^2}{R} \frac{\partial}{\partial R} \left[ K_{HV} \left( \frac{R \partial \hat{U}_R}{m \partial z} + R \frac{\partial \hat{U}_z}{\partial R} \right) \right] - \\ &\quad - \frac{m}{R} \frac{\partial}{\partial \alpha} \left( K_{HV} \frac{\partial \hat{U}_\alpha}{\partial z} \right) - \frac{m^2}{R^2} \frac{\partial}{\partial \alpha} \left( K_{HV} \frac{\partial \hat{U}_z}{\partial \alpha} \right) - \\ &\quad - \frac{4}{3} \frac{\partial}{\partial z} \left( K_{HH} \frac{\partial \hat{U}_z}{\partial z} \right) + \frac{2m^2}{3R} \frac{\partial}{\partial z} \left[ K_{HH} \frac{\partial}{\partial R} \left( \frac{\hat{U}_R R}{m} \right) \right] + \\ &\quad + \frac{2m}{R} \frac{\partial}{\partial z} \left( K_{HH} \frac{\partial \hat{U}_\alpha}{\partial \alpha} \right)\end{aligned}$$

In Eq. (8-30) we have used the abbreviations:

$$\begin{aligned}\Delta_1 &= \frac{2m}{3} \left[ \frac{\partial}{\partial R} \left( m \frac{\partial \dots}{\partial R} \right) + \frac{m^2}{R} \left( \frac{\partial R/m}{\partial R} \right) \frac{\partial \dots}{\partial R} \right] + \\ &\quad + \frac{m^2}{R^2} \frac{\partial^2 \dots}{\partial \alpha^2}\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \left( 1 + 2m \frac{\partial R/m}{\partial R} \right) m \frac{\partial \dots}{\partial R} + m R \frac{\partial^2 \dots}{\partial R^2} + \\ &\quad + \frac{2m}{3R} \frac{\partial^2 \dots}{\partial \alpha^2}\end{aligned}$$

$$\begin{aligned}\Delta_3 &= \frac{1}{3} \left[ \left( 8 \frac{m^2}{R} \frac{\partial R/m}{\partial R} + R \frac{\partial m/R}{\partial R} \right) \frac{\partial \dots}{\partial \alpha} + \right. \\ &\quad \left. + 2m \frac{\partial^2 \dots}{\partial R \partial \alpha} \right]\end{aligned}$$

## 9 List of Symbols

### Quantities

$\vec{a}, \vec{b}, \vec{c}$  ... configuration of arbitrary unit vectors

$\delta$  unit tensor,  $\delta$  variation symbol at Eq. (8-4)

$\delta_{ij}$  Kronecker delta-symbol (components of  $\delta$ )

$\vec{e}$  unit vector for a defined coordinate direction

$\vec{e}_x, \vec{e}_y, \vec{e}_z$  unit vectors in x, y, z-direction

$\epsilon$  total energy per unit mass

$\epsilon_T$  kinetic energy per unit mass

$\epsilon_R$  rotational kinetic energy per unit mass

$\epsilon_I$  internal energy per unit mass

$E$  eddy-kinetic energy per unit mass

$\vec{I}_e$  flux density of energy  $\epsilon$

$\vec{I}_h$  heat flux density

$\mathbf{I}$  friction stress tensor

$\mathbf{I}^{0,s}$  symmetric part of  $\mathbf{I}$  with zero trace

$\mathbf{I}^{a,s}$  antisymmetric part of  $\mathbf{I}$

$\mathcal{I}$  isotropic part of  $\mathbf{I}$

$K_H, K_{HH}, K_{HV}$  axisymmetric shear viscosity-coefficients

$K_m$  isotropic shear-viscosity coefficient

$K_h$  heat conductivity

$K_h^H, K_h^V$  horizontal, vertical heat conductivity

$K_r$  isotropic coefficient of rotational friction

$K_r^H, K_r^V$  horizontal, vertical coefficient of rotational friction

$K_v$  coefficient of volume viscosity

$\vec{\lambda}$  unit vector of the axis of symmetry

$\lambda_i$  direction cosines of  $\vec{\lambda}$

$m$  factor of stereographic map projection

$\mathbf{P}$  stress tensor

$P_{ij}$  components of  $\mathbf{P}$

$\mathbf{P}^s, \mathbf{P}^{a,s}$  symmetric, antisymmetric part of  $\mathbf{P}$

$\vec{P}^a$  axial vector referring to  $\mathbf{P}^{a,s}$

$P_i^a$  components of  $\vec{P}^a$

$p$  pressure

$\bar{\Pi} = \text{mean total pressure} = \bar{p} + \frac{2}{3} \bar{S} E$

$\rho$  density

$\bar{\rho}$  average density

$\vec{r}$  position vector

$\vec{R}$  external angular momentum per unit mass

$\vec{S}$  internal angular momentum per unit mass

$S$  entropy per unit mass

$\bar{\sigma} = \bar{\rho} \frac{ds}{dt} |_{irr.}$  entropy production per unit volume

$T$  temperature

$\bar{T}$  mean temperature

$\Theta$  (average) inertia moment referring to intrinsic rotations

$\vec{u}$  velocity vector

$u, v, w$  cartesian components of  $\vec{u}$

$\hat{u}$  turbulent mean of  $\vec{u}$

$\vec{u}''$  velocity fluctuation

$\rho \vec{u}'' \vec{u}''$  Reynolds-stress

$\vec{\omega}$  (average) angular velocity of internal particle rotations

$\omega_i$  cartesian components of  $\vec{\omega}$

$\mathcal{W}$  antisymmetric tensor referring to  $\vec{\omega}$

### Operators

$\partial/\partial x_i$  partial derivative with respect to the space coordinates

$\partial/\partial t$  partial derivative with respect to time

$d/dt$  total derivative =  $\partial/\partial t + \vec{u} \cdot \nabla$

$D/Dt = \partial/\partial t + \hat{u} \cdot \nabla$

$x_i, \cdot, :$  vector, scalar, double-scalar product

$\nabla$  Del-operator

$\nabla \cdot$  space-divergence of a vector or a tensor

$\nabla \times$  rotation (or curl) of a vector

$\Sigma$  symbol designating a sum

$\text{Tr}$  symbol designating the trace of a tensor

### Coefficient indices

$H$  subscript referring to a horizontal surface

$M$  subscript referring to momentum

$h$  subscript referring to heat

$R, r$  subscript referring to rotation

$V$  subscript referring to vertical direction

$v$  subscript referring to volume

$X, Y, Z$  designating cartesian coordinates

$R, \alpha, z$  designating cylinder coordinates

$i, j, k$  integer indices; summation indices in connexion with  $\Sigma$

APPENDIX A

1 Balance Equations For External And Internal Angular Momentum

The equation of external angular momentum can be derived by vectorial multiplication of the equation of motion (Eq. 2-1) with the position vector  $\vec{r}$ , i. e.  $\vec{r} \times (\rho \frac{d\vec{u}}{dt} + \nabla \cdot \mathbf{P}) = 0$ . We find then

$$\begin{aligned} \rho \frac{d(\vec{r} \times \vec{u})}{dt} + \nabla \cdot (\vec{r} \times \mathbf{P}) - \nabla \cdot (\vec{r} \times \mathbf{P}) &= 0 \\ \text{"} + \text{"} + (\tilde{\mathbf{P}} \times \vec{r}) \cdot \nabla &= 0 \\ \text{"} + \text{"} + (\tilde{\mathbf{P}} \cdot \nabla) \times \vec{r} &= 0 \end{aligned}$$

$$\rho \frac{d(\vec{r} \times \vec{u})}{dt} + \nabla \cdot (\vec{r} \times \mathbf{P}) + (\tilde{\mathbf{P}} \cdot \nabla \vec{r})_X = 0 \quad (\text{A-1})$$

where  $(\tilde{\mathbf{P}} \cdot \nabla \vec{r})_X$  denotes the vector which is gained by external multiplication by the vector pair within the bracket. With well-known tensor relationships the vector of this bracket expression can still be formulated in terms of the total stress tensor, since

$$(\tilde{\mathbf{P}} \cdot \nabla \vec{r})_X = (\tilde{\mathbf{P}} \cdot \delta)_X = \tilde{\mathbf{P}}_X = -\mathbf{P}_X \quad (\text{A-2})$$

Note that only the asymmetric part of the stress contributes to the vector  $\mathbf{P}_X$ , hence

$$\begin{aligned} \mathbf{P}_X &= P_{12} \vec{e}_x \times \vec{e}_y + P_{13} \vec{e}_x \times \vec{e}_z - \\ &- P_{12} \vec{e}_y \times \vec{e}_x + P_{23} \vec{e}_y \times \vec{e}_z - \\ &- P_{13} \vec{e}_z \times \vec{e}_x - P_{23} \vec{e}_z \times \vec{e}_y = \\ &= 2(P_{23} \vec{e}_x + P_{13} \vec{e}_y + P_{12} \vec{e}_z) \end{aligned}$$

Thus we may alternatively write  $\mathbf{P}_X$  as the twofold of an axial vector  $\vec{P}^a$  with the components  $P_1^a = P_{23} = -P_{32}$ , cycl. With

$$\vec{P}^a = \frac{1}{2} \mathbf{P}_X \quad (\text{A-3})$$

we obtain together with Eq. (A-2) that the balance equation of external angular momentum (A-1) may finally be written as

$$\begin{aligned} \rho \frac{d\vec{R}}{dt} + \nabla \cdot (\vec{r} \times \mathbf{P}) &= 2\vec{P}^a, \\ \nabla \cdot (\vec{r} \times \mathbf{P}) &\rightarrow \sum_K \frac{\partial}{\partial x_K} (x_i P_{Kj} - x_j P_{Ki}) \end{aligned} \quad (\text{A-4})$$

The law of conservation of the total angular momentum reads

$$\rho \frac{d\vec{Q}}{dt} + \nabla \cdot (\vec{r} \times \mathbf{P}) = 0 \quad (\text{A-5})$$

where we have omitted according to de Groot and Mazur (1969) a divergence term due to intrinsic flux density.

The balance equation of the internal angular momentum follows from the difference of Eqs. (A-5) and (A-4) to

$$\rho \frac{d\vec{S}}{dt} = -2\vec{P}^a \quad (\vec{S} = \vec{Q} - \vec{R} = \theta \vec{\omega}) \quad (\text{A-6})$$

2 The Product  $\vec{u} \cdot \nabla \cdot \mathbf{P}$

This expression can be divided as follows

$$\begin{aligned} \vec{u} \cdot \nabla \cdot \mathbf{P} &= (\nabla \cdot \mathbf{P}) \cdot \vec{u} = \\ &= \nabla \cdot (\mathbf{P} \cdot \vec{u}) - (\mathbf{P} \cdot \vec{u}) \cdot \nabla \\ &= \nabla \cdot (\mathbf{P} \cdot \vec{u}) - \mathbf{P} : \vec{u} \nabla \\ &= \nabla \cdot (\mathbf{P} \cdot \vec{u}) - \tilde{\mathbf{P}} : \nabla \vec{u} \end{aligned} \quad (\text{A-7})$$

The first part represents the divergence term of kinetic energy and the second part the energy dissipation; both terms are effected by the total stress of the equation of motion.

APPENDIX B

Axisymmetric Tensors Of First To Fourth Order

Unit vector of the invariant axis of symmetry:

$$\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3), \quad \lambda_i = \text{Direction cosines}$$

Sketch-configuration:  $\vec{a}, \vec{b}, \vec{c}, \vec{d}$  etc., arbitrary unit vectors.

1 Tensor Of First Order (Vector):

$$\vec{K} = (K_1, K_2, K_3)$$

Scalar form:  $K(\vec{\lambda}; \vec{a}) = \sum K_i a_i$

Invariance condition:  $K(\vec{\lambda}; \vec{a}) = K(\vec{\lambda}; \vec{a}')$

Scalar fundamental invariants for polar vectors:  $\vec{\lambda} \cdot \vec{a}$ ,  $\vec{\lambda} \cdot \vec{\lambda}$ ,  $\vec{a} \cdot \vec{a}$  and homogeneous combinations of them.

Consider that  $K(\vec{\lambda}; \vec{a})$  is linear and homogeneous with respect to  $a_i$  which is only satisfied by  $\vec{\lambda} \cdot \vec{a}$ . Hence

$$K(\vec{\lambda}; \vec{a}) = K \vec{\lambda} \cdot \vec{a} = \sum K \lambda_i a_i \quad (B-1)$$

$$\vec{K} = K \vec{\lambda} \quad \text{or} \quad K_i = K \lambda_i \quad (B-2)$$

This means that one scalar is needed to describe the three vector components  $K_i$ . Special case:  $\vec{\lambda} = (0, 0, 1)$ , invariant direction coincides with the  $Z$ -axis of coordinates, which yields  $\vec{K} = (0, 0, K)$ .

Axial vectors must be zero since with the vectors  $\vec{\lambda}$  and  $\vec{a}$  no invariant of type (b) of Eq. (4-4) can be constructed (see also Sect. 4, formula (4-7)).

2 Tensors Of Second Order (Dyads):

$$K_{ij} \quad (i, j = 1, 2, 3)$$

Scalar form:  $K(\vec{\lambda}; \vec{a}, \vec{b}) = \sum K_{ij} a_i b_j$

Invariance condition:  $K(\vec{\lambda}; \vec{a}, \vec{b}) = K(\vec{\lambda}; \vec{a}', \vec{b}')$

( $\alpha$ ) Polar dyads

Scalar fundamental invariants:  $\vec{\lambda} \cdot \vec{\lambda}$ ,  $\vec{\lambda} \cdot \vec{a}$ ,  $\vec{\lambda} \cdot \vec{b}$ ,  $\vec{a} \cdot \vec{a}$ ,  $\vec{a} \cdot \vec{b}$ ,  $\vec{b} \cdot \vec{b}$  and homogeneous combinations of them.

The function  $K(\vec{\lambda}; \vec{a}, \vec{b})$  is linear and homogeneous with respect to  $a_i b_j$  on account of which  $K(\vec{\lambda}; \vec{a}, \vec{b})$  could at most depend upon  $(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})$  and  $(\vec{a} \cdot \vec{b})$ .

This means that

$$K(\vec{\lambda}; \vec{a}, \vec{b}) = A \vec{\lambda} \vec{\lambda} : \vec{a} \vec{b} + B \delta : \vec{a} \vec{b} = (A \vec{\lambda} \vec{\lambda} + B \delta) : \vec{a} \vec{b} = \sum (A \lambda_i \lambda_j + B \delta_{ij}) a_i b_j \quad (B-3)$$

and

$$K = A \vec{\lambda} \vec{\lambda} + B \delta, \quad \text{or} \quad K_{ij} = A \lambda_i \lambda_j + B \delta_{ij} \quad (B-4)$$

Hence  $K_{ij} = K_{ji}$ , symmetrical tensor, whose nine components are determined by two constants  $A$  and  $B$ . Equation (B-4) and the condition of vanishing trace yield together

$$\sum K_{ii} = 0 : 0 = \sum A \lambda_i^2 + 3B$$

and thus  $B = -\frac{A}{3} \rightarrow$

$$K_{ij} = A(\lambda_i \lambda_j - \frac{1}{3} \delta_{ij}), \quad \text{or} \quad K = A(\vec{\lambda} \vec{\lambda} - \frac{1}{3} \delta) \quad (B-5)$$

( $\beta$ ) Axial dyads

Scalar fundamental invariant:  $\vec{\lambda} \cdot (\vec{a} \times \vec{b})$  with which we obtain for the scalar form

$$K(\vec{\lambda}; \vec{a}, \vec{b}) = P \vec{\lambda} \cdot (\vec{a} \times \vec{b}) = \sum P \epsilon_{ijk} \lambda_k a_i b_j$$

showing that the form depends linearly on  $a_i b_j$ . Thus we find

$$K_{ij} = P \sum_k \epsilon_{ijk} \lambda_k \quad (B-6)$$

We insert into Eq. (B-6) the values of the permutation tensor  $\epsilon_{ijk}$ , i. e.

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ ,  $\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$  all other elements are zero, and obtain  $K_{ij}$  in matrix as well as extensive notation as follows

$$K_{ij} = P \begin{pmatrix} 0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & 0 \end{pmatrix}, \quad \text{or}$$

$$K = P \left\{ \lambda_3 (\vec{e}_x \vec{e}_y - \vec{e}_y \vec{e}_x) + \lambda_2 (\vec{e}_z \vec{e}_x - \vec{e}_x \vec{e}_z) + \lambda_1 (\vec{e}_y \vec{e}_z - \vec{e}_z \vec{e}_y) \right\} \quad (B-7)$$

It is easily seen after external multiplication within the pairs of unit vectors of Eq. (B-7) that  $K$  can alternatively be expressed by the vector

$$\frac{1}{2} K_x = P \vec{\lambda} = P(\lambda_1, \lambda_2, \lambda_3) \quad (B-8)$$

( $\gamma$ ) If the invariant direction is identical with the  $Z$ -axis of coordinates we have  $\vec{\lambda} = (0, 0, 1)$ . Then polar tensors  $K_{ij}$  are in accordance with Eqs. (B-4) and (B-5) diagonal of the

form

$$K_{ij} = \begin{pmatrix} B & 0 \\ 0 & B+A \end{pmatrix}, \text{ or } K_{ij} = \begin{pmatrix} -\frac{A}{3} & 0 \\ 0 & -\frac{A}{3} \end{pmatrix}$$

while the axial version of Eq. (B-8) may be represented by the vertically directed vector

$$\frac{1}{2} K_x = \mathcal{P} \vec{e}_z$$

### 3 Tensors of Third Order:

$$K_{ijk} \quad (i, j, k = 1, 2, 3)$$

Scalar form:  $K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}) = \sum K_{ijk} a_i b_j c_k$

Invariance condition:

$$K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}) = K(\vec{\lambda}; \vec{a}', \vec{b}', \vec{c}')$$

#### (α) Polar tensors

Scalar fundamental invariants:  $\vec{\lambda} \cdot \vec{\lambda}, \vec{\lambda} \cdot \vec{a}, \vec{\lambda} \cdot \vec{b}, \vec{\lambda} \cdot \vec{c}, \vec{a} \cdot \vec{a}, \vec{a} \cdot \vec{b}, \vec{a} \cdot \vec{c}, \vec{b} \cdot \vec{b}, \vec{b} \cdot \vec{c}, \vec{c} \cdot \vec{c}$  as well as homogeneous combinations of them.

The form  $K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c})$  depends linearly and homogeneously upon  $a_i, b_j, c_k$  on account of which only the four combinations  $(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c}), (\vec{\lambda} \cdot \vec{a})(\vec{b} \cdot \vec{c}), (\vec{\lambda} \cdot \vec{b})(\vec{a} \cdot \vec{c}), (\vec{\lambda} \cdot \vec{c})(\vec{a} \cdot \vec{b})$  are to be taken into consideration. Thus we may write

$$\begin{aligned} K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}) &= A(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c}) + \\ &+ B(\vec{\lambda} \cdot \vec{a})(\vec{b} \cdot \vec{c}) + C(\vec{\lambda} \cdot \vec{b})(\vec{a} \cdot \vec{c}) + \\ &+ D(\vec{\lambda} \cdot \vec{c})(\vec{a} \cdot \vec{b}) = \sum_{i,j,k} (A\lambda_i \lambda_j \lambda_k + \\ &+ B\lambda_i \delta_{jk} + C\lambda_j \delta_{ik} + D\lambda_k \delta_{ij}) a_i b_j c_k \end{aligned} \quad (B-9)$$

and

$$\begin{aligned} K_{ijk} &= A\lambda_i \lambda_j \lambda_k + B\lambda_i \delta_{jk} + \\ &+ C\lambda_j \delta_{ik} + D\lambda_k \delta_{ij} \end{aligned} \quad (B-10)$$

This formula shows that the 27 components of  $K_{ijk}$  are determined by four differing constants if the indices are not subjected to further conditions.

#### Consequences of the symmetric conditions (3-11)

$$(a) \quad K_{ijk} = K_{jik} \quad \text{and} \quad K_{ijk} = K_{ikj} :$$

$$\begin{aligned} A\lambda_i \lambda_j \lambda_k + B\lambda_i \delta_{jk} + C\lambda_j \delta_{ik} + D\lambda_k \delta_{ij} = \\ A\lambda_i \lambda_j \lambda_k + B\lambda_j \delta_{ik} + C\lambda_i \delta_{jk} + D\lambda_k \delta_{ji} \rightarrow \end{aligned}$$

$$\begin{aligned} \rightarrow C=B; \quad A\lambda_i \lambda_j \lambda_k + B(\lambda_i \delta_{jk} + \lambda_j \delta_{ik}) \\ + D\lambda_k \delta_{ij} = A\lambda_i \lambda_k \lambda_j + B(\lambda_i \delta_{kj} + \lambda_k \delta_{ij}) \\ + D\lambda_j \delta_{ik} \rightarrow \underline{B=C=0, D=0} \end{aligned}$$

(b)  $\sum_i K_{iik} = 0$  i. e. zero trace with respect to the first pair of indices. From (a) thus follows that

$$A \sum \lambda_i^2 \lambda_k = A \lambda_k \sum \lambda_i^2 = A \lambda_k = 0$$

which is satisfied for each direction cosine  $\lambda_k$  if  $A = 0$ . Together with the results of section 5 a we obtain then

$$K_{ijk} = 0 \quad (B-11)$$

Polar axisymmetric exchange tensors of third order must therefore disappear under these conditions of symmetry.

#### (β) Axial tensors

Scalar fundamental invariant:  $\vec{a} \cdot (\vec{b} \times \vec{c})$  with which the scalar form can be formulated by

$$\begin{aligned} K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}) &= Q \vec{a} \cdot (\vec{b} \times \vec{c}) = \\ &= \sum Q \epsilon_{ijk} a_i b_j c_k \end{aligned}$$

as linear and homogeneous function of the expression  $a_i b_j c_k$ . Hence

$$\left. \begin{aligned} K_{ijk} &= Q \epsilon_{ijk} \quad \text{with} \\ \epsilon_{ijk} &= \begin{cases} \epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1 \\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \\ \epsilon_{iik} = \epsilon_{ikk} = \epsilon_{iji} = 0 \end{cases} \end{aligned} \right\} (B-12)$$

Since the permutation tensor  $\epsilon_{ijk}$  is completely antisymmetric, the tensor  $K_{ijk}$  must also be antisymmetric in all its 27 components.

Thus, subjecting  $K_{ijk}$  of Eq. (B-12) to any symmetry relation quoted in sect. 3, e. g.

$K_{ijk} = K_{jik}$  or  $K_{ijk} = K_{ikj}$  the necessary and sufficient condition to be satisfied is

$$Q = 0, \quad \text{i. e. } K_{ijk} = 0 \quad (B-13)$$

which shows that axisymmetrically axial tensors of third order must also vanish under any index symmetry.

4 Tensors Of Fourth Order:

$$K_{ijkl} \quad (i, j, k, l = 1, 2, 3)$$

Scalar form:

$$K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d}) = \sum K_{ijkl} a_i b_j c_k d_l$$

Invariance condition:

$$K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d}) = K(\vec{\lambda}; \vec{a}', \vec{b}', \vec{c}', \vec{d}')$$

Scalar fundamental invariants:

$$\vec{\lambda} \cdot \vec{\lambda}, \vec{\lambda} \cdot \vec{a}, \vec{\lambda} \cdot \vec{b}, \vec{\lambda} \cdot \vec{c}, \vec{\lambda} \cdot \vec{d}, \vec{a} \cdot \vec{a}, \vec{a} \cdot \vec{b}, \vec{a} \cdot \vec{c}, \vec{a} \cdot \vec{d}, \vec{b} \cdot \vec{b}, \vec{b} \cdot \vec{c}, \vec{b} \cdot \vec{d}, \vec{c} \cdot \vec{c}, \vec{c} \cdot \vec{d}, \vec{d} \cdot \vec{d}$$

with which 10  $\vec{a}, \vec{b}, \vec{c}, \vec{d}$  - linear combinations

$$\begin{aligned} &(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c})(\vec{\lambda} \cdot \vec{d}), (\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{c} \cdot \vec{d}), \\ &(\vec{\lambda} \cdot \vec{c})(\vec{\lambda} \cdot \vec{d})(\vec{a} \cdot \vec{b}), (\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{c})(\vec{b} \cdot \vec{d}), (\vec{\lambda} \cdot \vec{b}) \cdot \\ &\cdot (\vec{\lambda} \cdot \vec{c})(\vec{a} \cdot \vec{d}), (\vec{\lambda} \cdot \vec{d})(\vec{\lambda} \cdot \vec{a})(\vec{b} \cdot \vec{c}), (\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{d}) \cdot \\ &\cdot (\vec{a} \cdot \vec{c}), (\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d}), (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}), (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \end{aligned}$$

can be formed needed to determine the axisymmetric form  $K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d})$ . In analogy to previous equations we then have:

$$\begin{aligned} K(\vec{\lambda}; \vec{a}, \vec{b}, \vec{c}, \vec{d}) &= A(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c})(\vec{\lambda} \cdot \vec{d}) + \\ &+ B(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{b})(\vec{c} \cdot \vec{d}) + C(\vec{\lambda} \cdot \vec{c})(\vec{\lambda} \cdot \vec{d})(\vec{a} \cdot \vec{b}) + \\ &+ D(\vec{\lambda} \cdot \vec{a})(\vec{\lambda} \cdot \vec{c})(\vec{b} \cdot \vec{d}) + E(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{c})(\vec{a} \cdot \vec{d}) + \\ &+ F(\vec{\lambda} \cdot \vec{d})(\vec{\lambda} \cdot \vec{a})(\vec{b} \cdot \vec{c}) + G(\vec{\lambda} \cdot \vec{b})(\vec{\lambda} \cdot \vec{d})(\vec{a} \cdot \vec{c}) + \\ &+ H(\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d}) + I(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) + K(\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) = \\ &= \sum_{i,j,k,l} \left\{ A \lambda_i \lambda_j \lambda_k \lambda_l + B \lambda_i \lambda_j \delta_{kl} + \right. \\ &+ C \lambda_k \lambda_l \delta_{ij} + D \lambda_i \lambda_k \delta_{jl} + E \lambda_j \lambda_k \delta_{il} + \\ &+ F \lambda_i \lambda_l \delta_{jk} + G \lambda_j \lambda_l \delta_{ik} + H \delta_{ij} \delta_{kl} + \\ &\left. + I \delta_{ik} \delta_{jl} + K \delta_{il} \delta_{jk} \right\} a_i b_j c_k d_l \end{aligned} \quad (B-14)$$

from which follows

$$\begin{aligned} K_{ijkl} &= A \lambda_i \lambda_j \lambda_k \lambda_l + B \lambda_i \lambda_j \delta_{kl} + (B-15) \\ &+ C \lambda_k \lambda_l \delta_{ij} + D \lambda_i \lambda_k \delta_{jl} + E \lambda_j \lambda_k \delta_{il} + \\ &+ F \lambda_i \lambda_l \delta_{jk} + G \lambda_j \lambda_l \delta_{ik} + H \delta_{ij} \delta_{kl} + \\ &+ I \delta_{ik} \delta_{jl} + K \delta_{il} \delta_{jk} \end{aligned}$$

All 81 components of the axisymmetric tensor of 4th order are completely described by means of the 10 scalars A, B, ..., K.

Further index conditions due to Eq. (3-12)

(a) Symmetry with respect to both pairs of indices:

$$K_{ijkl} = K_{jike} = K_{ijek} \quad (B-16)$$

It is obvious that the terms connected with the parameters A, B, C and H in Eq. (B-15) are a priori symmetrical in both pairs of indices. Consequently, condition (B-16) acts restrictively only on the remaining 6 expressions. Thereby one is easily convinced that, in order to satisfy Eq. (B-15),  $D = E = F = G$  and  $I = K$  must be valid, on account of which  $K_{ijkl}$  still depends on six different scalars, i. e.

$$\begin{aligned} K_{ijkl} &= A \lambda_i \lambda_j \lambda_k \lambda_l + B \lambda_i \lambda_j \delta_{kl} + (B-17) \\ &+ C \lambda_k \lambda_l \delta_{ij} + D(\lambda_i \lambda_k \delta_{jl} + \lambda_i \lambda_k \delta_{il} + \\ &+ \lambda_i \lambda_l \delta_{jk} + \lambda_j \lambda_l \delta_{ik}) + H \delta_{ij} \delta_{kl} + \\ &+ I(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \end{aligned}$$

(b)  $\sum_i K_{iikl} = 0$  (vanishing trace for the first pair of indices).

As both equations (B-15) and (B-17) are independent of the direction and the value of the invariant axis  $\vec{\lambda}$ , we set for convenience sake  $\vec{\lambda} = (0, 0, 1)$ , i. e. the vector of the invariant axis is assumed to coincide with the unit vector of the vertical direction of a cartesian system of coordinates.

Alternative cases:

$$(\alpha) \text{ if } k = l = 1 \text{ or } 2 \rightarrow B + 3H + 2I = 0 \quad (B-18)$$

$$(\beta) \text{ if } k = l = 3 \rightarrow A + 3C + 4D = 0 \quad (B-19)$$

(γ) combinations for  $k \neq l$  do not imply further conditions for the 6 scalars.

(c)  $\sum_k K_{ijkk} = 0$  (vanishing trace for the second pair of indices)

We set again, according to (b),  $\lambda_1 = \lambda_2 = 0$ ,  $\lambda_3 = 1$  ( $\lambda_i = \delta_{i3}$ )

Alternative cases:

( $\alpha$ ) if  $i = j = 1$  or  $2 \rightarrow C + 3H + 2 = 0$  (B-20)

( $\beta$ ) if  $i = j = 3 \rightarrow A + 3B + 4D = 0$  (B-21)

( $\gamma$ ) combinations for  $i \neq j$  do not imply further conditions for the 6 scalars.

Equations (B-18) to (B-21) yield

$$C = B, \quad I = -\frac{B}{2} - \frac{3}{2}H, \quad A = -(3B + 4D) \quad (B-22)$$

with which Eq. (B-17) then obtains the form

$$\begin{aligned} K_{ijkl} = & -(3B + 4D)\lambda_i\lambda_j\lambda_k\lambda_l + \quad (B-23) \\ & + B(\lambda_i\lambda_j\delta_{kl} + \lambda_k\lambda_l\delta_{ij}) + D(\lambda_i\lambda_k\delta_{jl} + \\ & + \lambda_j\lambda_k\delta_{il} + \lambda_i\lambda_l\delta_{jk} + \lambda_j\lambda_l\delta_{ik}) + H\delta_{ij}\delta_{kl} \\ & - \frac{1}{2}(B + 3H)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned}$$

All 81 components of the fourth-ranked tensor  $K$  are expressible in terms of three differing scalars  $B$ ,  $D$  and  $H$ . Under the given forced conditions of symmetry Eq. (B-23) represents the most general and simplest form of  $K_{ijkl}$ .

With regard to practical applications of Eq. (B-23) i. e. in order to investigate eddy-frictional processes in the atmosphere and the ocean (see Sect. 4, 5 and 6 as well as Kamenskovich, 1967, Monin and Zilitinkevich, 1968) we will introduce, instead of the coefficients  $B$ ,  $D$  and  $H$ , the coefficients  $K_H, K_{HH}, K_{HV}$  through the following relations

$$K_H = -(B + 3H), \quad K_{HH} = -3(B + H),$$

$$K_{HV} = -2D - (B + 3H), \quad \text{or} \quad (B-24)$$

$$B = \frac{K_H - K_{HH}}{2}, \quad D = \frac{K_{HV} - K_H}{2}, \\ H = \frac{K_{HH}}{3} - \frac{K_H - K_{HH}}{2}$$

Then Eq. (B-23) can be written in terms of the  $K$ -parameters as follows:

$$\begin{aligned} K_{ijkl} = & \frac{K_H}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \quad (B-25) \\ & - \frac{K_{HH}}{3}\delta_{ij}\delta_{kl} + \\ & + \frac{K_H - K_{HH}}{2}(\lambda_i\lambda_j\delta_{kl} + \lambda_k\lambda_l\delta_{ij} - \\ & - \delta_{ij}\delta_{kl}) \\ & + \frac{K_{HV} - K_H}{2}(\lambda_i\lambda_k\delta_{jl} + \lambda_j\lambda_k\delta_{il} + \\ & + \lambda_i\lambda_l\delta_{jk} + \lambda_j\lambda_l\delta_{ik}) + \\ & + \frac{1}{2}(K_H + 3K_{HH} - 4K_{HV})\lambda_i\lambda_j\lambda_k\lambda_l \end{aligned}$$

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